NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2556

BUCKLING OF RECTANGULAR SANDWICH PLATES SUBJECTED TO EDGEWISE COMPRESSION WITH LOADED EDGES SIMPLY SUPPORTED AND UNLOADED EDGES CLAMPED

By Kuo Tai Yen, V. L. Salerno, and N. J. Hoff
Polytechnic Institute of Brooklyn

NACA
Washington
January 1952

1E

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2556

BUCKLING OF RECTANGULAR SANDWICH PLATES SUBJECTED TO EDGEWISE COMPRESSION WITH LOADED EDGES SIMPLY SUPPORTED AND UNLOADED EDGES CLAMPED

By Kuo Tai Yen, V. L. Salerno, and N. J. Hoff

SUMMARY

The compressive stress for buckling is calculated for a rectangular flat sandwich plate with loaded edges simply supported and unloaded edges rigidly clamped. In the calculations Hoff's differential equations are integrated by Leggett's method in order to obtain lower bounds and by Galerkin's method to establish upper bounds. The true values of the buckling stress are estimated as the arithmetic means of these bounds and are presented in a diagram which covers the entire practical range of the geometric and mechanical quantities involved. The theoretical results are in satisfactory agreement with results of tests carried out at the Forest Products Laboratory.

INTRODUCTION

The expression "sandwich plate" designates a composite plate consisting of two thin faces and a thick core. In airplane construction the faces are usually of aluminum alloy and the core is of some lightweight material such as an expanded plastic or balsa wood. In the latter case the fibers of the wood are, in general, arranged perpendicularly to the plane of the plate. Since the modulus of elasticity of such a core in the plane of the plate is about one-thousandth of that of the faces, the normal stresses in the core are of little importance in resisting bending moments even though the usual ratio of face thickness to core thickness is between one-tenth and one-hundredth. On the other hand, the core performs a task in transmitting shear forces and undergoes considerable shearing deformations because its modulus of shear is low. Hence shearing deformations must not be disregarded in the analysis of sandwich plates.

In an earlier paper (reference 1) differential equations were derived for rectangular sandwich plates subjected to transverse and edgewise loading and in the derivation the finite bending rigidity of the individual faces was duly considered. The differential equations were integrated for compressive loading perpendicular to one pair of edges when all four edges of the plate were simply supported, and the buckling stresses obtained were presented in diagrams. In the present report this work is continued and buckling stresses are calculated for compressive loads acting perpendicular to one pair of edges when the loaded edges are simply supported and the unloaded edges rigidly clamped. In order to obtain a close approximation to the true values of the buckling stress and to establish rigorously the accuracy of the solution, both lower and upper bounds were determined for the buckling stress. The former were obtained by Leggett's approach and the latter by Galerkin's method.

Although the calculations are somewhat complex, the final results are presented in a diagram (fig. 7) which can be used easily by the airplane designer. The few simple formulas needed in conjunction with the diagram are collected and their use is shown under the heading "Numerical Examples."

The calculations presented here were carried out at the Polytechnic Institute of Brooklyn under the sponsorship and with the financial aid of the National Advisory Committee for Aeronautics.

SYMBOLS

a_o, b_o, d_o — parameters depending on geometry and elastic properties of sandwich plate and on n

c — core thickness, inches

$c_o = (c + t)/L_x$

C — critical stress factor

D — bending rigidity of thin plate, pound-inches squared per inch

D_o — bending rigidity of sandwich plate, pound-inches squared per inch

E — Young's modulus

F — form factor of sandwich plate

G — shear modulus of face, psi

G_c	shear modulus of core, psi
k	reciprocal of aspect ratio of sandwich plate $\left(L_x/L_y\right)$
k_s	eigenvalues for vibration of a beam
$k_{so} = k_s/\pi$	
L_x,L_y	edge lengths of sandwich plate, inches
n	number of half waves in direction of loading
R_1,R_2	parameters depending on elastic properties of sandwich plate
R	sandwich stiffness parameter
$r = c/t$	
t	face thickness, inches
u,v,w	components of displacements in direction of x-, y-, and z-axes, respectively
x,y,z	rectangular coordinates
Y	compressive edge load in direction of y-axis, pounds per inch
Y_o	nondimensional form of compressive edge load in direction of y-axis
δ_{mn}	Kronecker delta
Δ	Laplacian operator
ξ,η,ζ	nondimensional coordinate axes $\left(\xi = \frac{x}{L_x};\ \eta = \frac{y}{L_y};\ \zeta = 2\xi - 1\right)$
μ	Poisson's ratio
$\sigma_{cr,o}$	buckling stress of two independent faces, psi

σ_{cr} buckling stress of sandwich plate, psi

λ_1, λ_2 parameters depending on elastic properties of sandwich plate

$\lambda = 4Fc$

SOLUTION BY LEGGETT'S METHOD

Figure 1 shows the sandwich plate and its loading. The equilibrium conditions of the plate were derived in reference 1 from the essential parts of the strain energy and the potential of the external loads with the aid of the principle of virtual displacements. They can be given in the following form:

$$-\frac{Et}{1-\mu^2}\left[2u_{xx} + (1-\mu)u_{yy} + (1+\mu)v_{xy}\right] + \frac{2G_c{}^2}{c+t}\left(\frac{2u}{c+t} + w_x\right) = 0 \qquad (1)$$

$$-\frac{Et}{1-\mu^2}\left[2v_{yy} + (1-\mu)v_{xx} + (1+\mu)u_{xy}\right] + \frac{2G_c{}^2}{c+t}\left(\frac{2v}{c+t} + w_y\right) = 0 \qquad (2)$$

$$\frac{Et^3}{6\left(1-\mu^2\right)}\Delta^2 w - \frac{2G_c c}{c+t}\left(u_x + v_y\right) + Yw_{yy} - G_c c\,\Delta w = 0 \qquad (3)$$

where u, v, and w are displacements in the x-, y-, and z-directions, respectively, of points in the middle plane of the upper face plate of the sandwich, while those in the lower face are $-u$, $-v$, and w, as shown in figures 2 and 3. The subscripts x and y denote differentiation with respect to the coordinates x and y. Since a right-hand coordinate system is used in this report, the signs of the last term in the first two equations and the second term of the third equation are opposite to those given in reference 1.

For a sandwich plate simply supported at the two loaded edges $y = 0$ and $y = L_y$ and clamped at the edges $x = 0$ and $x = L_x$ the boundary conditions are:

$$u = 0 \qquad \text{at all four edges} \tag{4a}$$

$$v = 0 \qquad \text{at} \quad x = 0, L_x \tag{4b}$$

$$v_y + \mu u_x = 0 \qquad \text{at} \quad y = 0, L_y \tag{4c}$$

$$w_x = 0 \qquad \text{at} \quad x = 0, L_x \tag{4d}$$

$$w_{yy} = 0 \qquad \text{at} \quad y = 0, L_y \tag{4e}$$

$$w = 0 \qquad \text{at all four edges} \tag{4f}$$

If the new variables

$$\xi = \frac{x}{L_x} \qquad \eta = \frac{y}{L_y} \qquad k = \frac{L_x}{L_y}$$

are introduced and the notation

$$R_1 = \frac{G_c c}{D_o} L_x^2 \qquad R_2 = \frac{G_c c}{2D} L_x^2 \qquad c_o = \frac{c + t}{L_x} \qquad Y_o = \frac{Y}{2D} L_x^2$$

is used where

$$D_o = \frac{Et(c + t)^2}{2(1 - \mu^2)} \qquad D = \frac{Et^3}{12(1 - \mu^2)}$$

equations (1), (2), and (3) can be written in the following nondimensional form:

$$-\left[2u_{\xi\xi} + (1 - \mu)k^2 u_{\eta\eta} + (1 + \mu)kv_{\xi\eta}\right] + 2R_1 u + R_1 c_o w_\xi = 0 \tag{5}$$

$$-\left[2k^2 v_{\eta\eta} + (1 - \mu)v_{\xi\xi} + (1 + \mu)ku_{\xi\eta}\right] + 2R_1 v + R_1 c_o w_\eta = 0 \tag{6}$$

$$w_{\xi\xi\xi\xi} + 2k^2 w_{\xi\xi\eta\eta} + k^4 w_{\eta\eta\eta\eta} - \frac{2R_2}{c_o}\left(u_\xi + kv_\eta\right) +$$

$$Y_o k^2 w_{\eta\eta} - R_2\left(w_{\xi\xi} + k^2 w_{\eta\eta}\right) = 0 \tag{7}$$

The solution may be assumed to have the form:

$$u = \sum_{n=1}^{\infty} F_n(\xi) \sin n\pi\eta \tag{8a}$$

$$v = \sum_{n=1}^{\infty} G_n(\xi) \cos n\pi\eta \tag{8b}$$

$$w = \sum_{n=1}^{\infty} H_n(\xi) \sin n\pi\eta \tag{8c}$$

These functions satisfy all the boundary conditions if

$$F_n(\xi) = 0 \qquad \text{at} \quad \xi = 0,1 \tag{9}$$

$$G_n(\xi) = 0 \qquad \text{at} \quad \xi = 0,1 \tag{10}$$

$$H_n(\xi) = H_n'(\xi) = 0 \qquad \text{at} \quad \xi = 0,1 \tag{11}$$

Insertion of u, v, and w into equations (5), (6), and (7) yields:

$$-2F_n''(\xi) + \left[(1 - \mu)(kn)^2\pi^2 + 2R_1\right]F(\xi) + (1 + \mu)(kn)\pi G_n'(\xi) +$$

$$R_1 c_o H_n'(\xi) = 0 \tag{12}$$

$$-(1 + \mu)(kn)\pi F_n'(\xi) - (1 - \mu)G_n''(\xi) + 2\left[(kn)^2\pi^2 + R_1\right]G_n(\xi) +$$

$$(kn)\pi R_1 c_o H_n(\xi) = 0 \tag{13}$$

$$-\left(2R_2/c_o\right)F_n'(\xi) + \left(2R_2/c_o\right)(kn)\pi G_n(\xi) + H_n''''(\xi) - \left[2(kn)^2\pi^2 + R_2\right]H_n''(\xi) +$$

$$(kn)^2\pi^2\left[(kn)^2\pi^2 + R_2 - Y_o\right]H_n(\xi) = 0 \tag{14}$$

These three ordinary simultaneous differential equations can be solved exactly, but the algebraic manipulations become so cumbersome that an approximate method is preferable. The method used here was suggested by Leggett (reference 2), and was applied by Smith (reference 3) for the calculation of buckling loads of plywood plates.

The method consists in expressing the derivative of highest order $H_n''''(\xi)$ by means of a Fourier series, and then integrating the series term by term:

$$H_n''''(\xi) = \sum_{m=1}^{\infty} A_m(m\pi)^4 \sin m\pi\xi \tag{15a}$$

$$H_n'''(\xi) = \sum_{m=1}^{\infty} (-1)A_m(m\pi)^3 \cos m\pi\xi + 6A \tag{15b}$$

$$H_n''(\xi) = \sum_{m=1}^{\infty} (-1)A_m(m\pi)^2 \sin m\pi\xi + 6A\xi + 2B \tag{15c}$$

$$H_n'(\xi) = \sum_{m=1}^{\infty} A_m(m\pi) \cos m\pi\xi + 3A\xi^2 + 2B\xi + C \tag{15d}$$

$$H_n(\xi) = \sum_{m=1}^{\infty} A_m \sin m\pi\xi + A\xi^3 + B\xi^2 + C\xi + D \tag{15e}$$

The arbitrary constants A, B, C, and D are determined from the boundary conditions (equation (11)):

$$A = -\pi \sum_{m=1}^{\infty} mA_m \left[1 + (-1)^m\right] \tag{16a}$$

$$B = \pi \sum_{m=1}^{\infty} mA_m \left[2 + (-1)^m\right] \tag{16b}$$

$$C = -\pi \sum_{m=1}^{\infty} mA_m \tag{16c}$$

$$D = 0 \tag{16d}$$

Upon substitution of the expressions for $H_n'(\xi)$ and $H_n(\xi)$ from equations (15d) and (15e), equations (12) and (13) become:

$$F_n''(\xi) - K_1 F_n(\xi) - K_2 G_n'(\xi) = K_3 \left[\sum_{m=1}^{\infty} A_m(m\pi) \cos m\pi\xi + 3A\xi^2 + 2B\xi + C\right] \tag{17}$$

$$F_n'(\xi) + K_4 G_n''(\xi) - K_5 G_n(\xi) = K_6 \left[\sum_{m=1}^{\infty} A_m \sin m\pi\xi + A\xi^3 + B\xi + C\xi\right] \tag{18}$$

where

$$K_1 = \frac{(1 - \mu)}{2}(kn)^2\pi^2 + R_1 \qquad K_2 = \frac{1 + \mu}{2}(kn)\pi \qquad K_3 = \frac{R_1 c_o}{2}$$

$$K_4 = \frac{(1 - \mu)}{(1 + \mu)(kn)\pi} \qquad K_5 = \frac{2\left[(kn)^2\pi^2 + R_1\right]}{(1 + \mu)(kn)\pi} \qquad K_6 = \frac{R_1 c_o}{(1 + \mu)}$$

The solution of equations (17) and (18) can be obtained by adding together the complementary and particular integrals. The homogeneous equations are

$$F_n''(\xi) - K_1F_n(\xi) - K_2G_n'(\xi) = 0 \tag{17a}$$

$$F_n'(\xi) + K_4G_n''(\xi) - K_5G_n(\xi) = 0 \tag{18a}$$

and the complementary solution may be assumed in the form

$$F_n(\xi) = B_1e^{\gamma\xi} \qquad G_n(\xi) = C_1e^{\gamma\xi}$$

where the values of γ are to be determined from the vanishing of the following determinant:

$$\begin{vmatrix} \gamma^2 - K_1 & -K_2\gamma \\ \gamma & K_4\gamma^2 - K_5 \end{vmatrix} = 0$$

Solution of this determinant after substitution of the values of K_1, K_2, K_4, and K_5 yields:

$$\gamma_1^2 = (kn)^2\pi^2 + \frac{2R_1}{1 - \mu} \tag{19}$$

$$\gamma_2^2 = (kn)^2\pi^2 + R_1 \tag{20}$$

Hence the complementary solution is

$$F_n(\xi) = B_1e^{\gamma_1\xi} + B_2e^{-\gamma_1\xi} + B_3e^{\gamma_2\xi} + B_4e^{-\gamma_2\xi} \tag{21a}$$

$$G_n(\xi) = C_1e^{\gamma_1\xi} + C_2e^{-\gamma_1\xi} + C_3e^{\gamma_2\xi} + C_4e^{-\gamma_2\xi} \tag{21b}$$

where the coefficients B and C are not independent, but satisfy the relations

$$\frac{B_1}{C_1} = -\frac{B_2}{C_2} = \frac{(kn)\pi}{\gamma_1} \qquad \frac{B_3}{C_3} = -\frac{B_4}{C_4} = \frac{\gamma_2}{(kn)\pi}$$

The particular integral of equations (17) and (18) may be assumed in the form

$$F_n(\xi) = \sum_{m=1}^{\infty} f_m \cos m\pi\xi + E_1\xi^2 + E_2\xi + E_3 \tag{22a}$$

$$G_n(\xi) = \sum_{m=1}^{\infty} g_m \sin m\pi\xi + I_1\xi^3 + I_2\xi^2 + I_3\xi + I_4 \tag{22b}$$

where, after setting $\left(R_1c_o/2\right) = p$, the coefficients are found to be

$$f_m = -m\pi pA_m \Big/ \left[(m\pi)^2 + \gamma_2^2\right]$$

$$g_m = -kn\pi pA_m \Big/ \left[(m\pi)^2 + \gamma_2^2\right]$$

$$E_1 = -3pA \big/ \gamma_2^2$$

$$E_2 = -2pB \big/ \gamma_2^2$$

$$E_3 = -\frac{6pA}{\gamma_2^4} - \frac{pC}{\gamma_2^2}$$

$$I_1 = \frac{p(kn)\pi E_1}{3} - \frac{(kn)\pi pA}{\gamma_2^2}$$

$$I_2 = \frac{kn\pi pE_2}{2} - \frac{kn\pi p^2 B}{\gamma_2^{\,2}}$$

$$I_3 = -\frac{kn\pi pC}{\gamma_2^{\,2}} = knp\pi E_3 - \frac{6kn\pi pA}{\gamma_2^{\,4}}$$

$$I_4 = \frac{knp\pi E_2}{\gamma_2^{\,2}} - \frac{2kn\pi pB}{\gamma_2^{\,4}}$$

From equations (21) and (22), the general solution of the differential equations (17) and (18) is

$$F_n(\xi) = D_1 \cosh \gamma_1\xi + D_2 \sinh \gamma_1\xi + D_3 \cosh \gamma_2\xi + D_4 \sinh \gamma_2\xi + \sum_{m=1}^{\infty} f_m \cos m\pi\xi + E_1\xi^2 + E_2\xi + E_3 \tag{23a}$$

$$G_n(\xi) = \frac{\gamma_1}{kn\pi}\left(D_2 \cosh \gamma_1\xi + D_1 \sinh \gamma_1\xi\right) + \frac{kn\pi}{\gamma_2}\left(D_4 \cosh \gamma_2\xi + D_3 \sinh \gamma_2\xi\right) + \sum_{m=1}^{\infty} g_m \sin m\pi\xi + I_1\xi^3 + I_2\xi^2 + I_3\xi + I_4 \tag{23b}$$

where D_1, D_2, D_3, and D_4 are arbitrary constants which can be determined from the remaining four boundary conditions (9) and (10):

$$D_1 = \frac{1}{Z}\left\{\frac{\gamma_1}{kn\pi}\left(\cosh \gamma_1 - \cosh \gamma_2\right)Z_1 - \left[\sinh \gamma_1 - \frac{\gamma_1\gamma_2}{(kn)^2\pi^2} \sinh \gamma_2\right]Z_2\right\}$$

$$D_2 = \frac{1}{Z}\left[\left(\cosh \gamma_1 - \cosh \gamma_2\right)Z_2 - \left(\frac{\gamma_1}{kn\pi} \sinh \gamma_1 - \frac{kn\pi}{\gamma_2} \sinh \gamma_2\right)Z_1\right]$$

$$D_3 = -D_1 - E_3 - \sum_{m=1}^{\infty} f_m$$

$$D_4 = -\frac{\gamma_1\gamma_2}{(kn)^2\pi^2} D_2 - \frac{\gamma_2}{kn\pi} I_4$$

and

$$Z = \frac{\gamma_1}{kn\pi}\left\{2\left(1 - \cosh \gamma_1 \cosh \gamma_2\right) + \left[\frac{(kn)^2\pi^2}{\gamma_1\gamma_2} + \frac{\gamma_1\gamma_2}{(kn)^2\pi^2}\right] \sinh \gamma_1 \sinh \gamma_2\right\}$$

$$Z_1 = \sum_{m=1}^{\infty} f_m\left[\cosh \gamma_2 - (-1)^m\right] + I_4 \frac{\gamma_2 \sinh \gamma_2}{kn\pi} - E_1 - E_2 + E_3\left(\cosh \gamma_2 - 1\right)$$

$$Z_2 = \frac{kn\pi}{\gamma_2}\left(E_3 + \sum_{m=1}^{\infty} f_m\right) \sinh \gamma_2 + I_4\left(\cosh \gamma_2 - 1\right) - I_1 - I_2 - I_3$$

Thus the functions $F_n(\xi)$, $G_n(\xi)$, and $H_n(\xi)$ satisfy the differential equations (12) and (13) and all the boundary conditions formulated in equations (9), (10), and (11). It remains to satisfy equation (14) by the assumed functions. To this end $F_n'(\xi)$, $G_n(\xi)$, $H_n''''(\xi)$, $H_n''(\xi)$, and $H_n(\xi)$ are expanded in sine series. The series are inserted into equation (14) and the sums of the coefficients of like terms are equated to zero. In this manner a system of homogeneous algebraic equations infinite in number and linear in the constants A_1, A_2 . . . is obtained. The equations yield nontrivial solutions if their determinant vanishes. This condition permits the calculation of the critical load. Since the determinant is of infinite order, successive approximations to the critical load can be obtained by solving subdeterminants of increasing order. It is shown in appendix A that this process yields critical values which approach the exact value from below. Hence Leggett's method gives a lower bound for the critical load.

The calculations are carried out with the aid of the following expansions:

$$\sinh \gamma_i\xi = -2\pi \sinh \gamma_i \sum_{s=1}^{\infty} \frac{(-1)^s s}{\gamma_i^2 + (s\pi)^2} \sin s\pi\xi$$

$$\cosh \gamma_i\xi = 2\pi \sum_{s=1}^{\infty} \frac{s}{\gamma_i^2 + (s\pi)^2}\left[1 - (-1)^s \cosh \gamma_i\right] \sin s\pi\xi$$

where $i = 1, 2$ and

$$\xi^3 = -\frac{2}{\pi}\sum_{s=1}^{\infty} \frac{(-1)^s}{s}\left[1 - \frac{6}{(s\pi)^2}\right] \sin s\pi\xi$$

$$\xi^2 = -\frac{2}{\pi}\sum_{s=1}^{\infty} \frac{1}{s}\left\{(-1)^s + \frac{2}{(s\pi)^2}\left[1 - (-1)^s\right]\right\} \sin s\pi\xi$$

$$\xi = -\frac{2}{\pi}\sum_{s=1}^{\infty} \frac{(-1)^s}{s} \sin s\pi\xi$$

$$1 = \frac{2}{\pi}\sum_{s=1}^{\infty} \frac{1}{s}\left[1 - (-1)^s\right] \sin s\pi\xi$$

Next the following notation is introduced:

$$\left.\begin{aligned}
\lambda_1 &= R_1/\pi^2 = (2/3)(c/t)\left\{1\Big/\left[1 + (c/t)\right]^2\right\}\left(G_c/\sigma_{cr,o}\right) \\
\lambda_2 &= R_2/\pi^2 = 2(c/t)\left(G_c/\sigma_{cr,o}\right) \\
\lambda &= Y_o/\pi^2 = Y_{cr}L_x^2/2\pi^2 D = 4\sigma_{cr}/\sigma_{cr,o}
\end{aligned}\right\} \qquad (24)$$

where

$$\sigma_{cr} = Y_{cr}/2t \tag{24a}$$

and

$$\sigma_{cr,o} = \pi^2 E t^2 / 3\left(1 - \mu^2\right) L_x^{\,2} \tag{24b}$$

The requirement that the sums of the coefficients of like terms vanish yields the following linear equation:

$$A_s s^4 - A_s\left(\lambda_1\lambda_2\right)\frac{(kn)^2 + s^2}{(kn)^2 + \lambda_1 + s^2} -$$

$$\frac{4\lambda_1\lambda_2}{(s\pi)^3}\left\{(-1)^s 3A - \left[1 - (-1)^s\right]\right\} B \frac{(kn)^2\left[(kn)^2 + \lambda_1\right] + \lambda_1 s^2}{\left[(kn)^2 + \lambda_1\right]^2} +$$

$$\left\{\left[(kn)^2 + \lambda_2\right]\left[(kn)^2 + s^2\right] + (kn)^2\left(s^2 - \lambda\right)\right\}\left\{A_s + \frac{4}{(s\pi)^3}\left[(-1)^s 3A -\right.\right.$$

$$\left.\left.\left(1 - (-1)^s\right)B\right]\right\} + \frac{2s\left(\lambda_1\lambda_2\right)}{(kn)^2 + \lambda_1 + s^2}\left[1 - \frac{(kn)^2}{(kn)^2 + \lambda_1}\right]H = 0 \tag{25}$$

$$s = 1, 2, \ldots$$

where

$$(-1)^s 3A - \left[1 - (-1)^s\right]B = -6\pi\left(2A_2 + 4A_4 + \ldots + 2mA_{2m} + \ldots\right)$$

when $s = 2m,\quad m = 1, 2, 3$

$$= -2\pi\left[A_1 + 3A_3 + \ldots + (2m - 1)A_{2m-1} + \ldots\right]$$

when $s = 2m - 1,\quad m = 1, 2$

$$H = \frac{1}{P_o}\left(P_1 + \frac{1}{\sinh \gamma_1} P_2 + \frac{1}{\cosh \gamma_2} P_3\right) \tag{26}$$

$$P_o = \gamma_2 \left\{ 2\left(\frac{1}{\sinh \gamma_1 \cosh \gamma_2} - \frac{\cosh \gamma_1}{\sinh \gamma_1} \right) + \left[\frac{(kn)^2\pi^2}{\gamma_1\gamma_2} + \frac{\gamma_1\gamma_2}{(kn)^2\pi^2} \right] \frac{\sinh \gamma_2}{\cosh \gamma_2} \right\}$$

$$P_1 = \left\{ \sum_{m=1}^{\infty} \frac{m^3 A_m \left[1 + (-1)^{m+s}\right]}{m^2 + (kn)^2 + \lambda_1} - \frac{6A\left[1 + (-1)^s\right]}{\gamma_2{}^2\pi} \right\} \left[\frac{\cosh \gamma_1 \sinh \gamma_2}{\sinh \gamma_1 \cosh \gamma_2} - \frac{\gamma_1\gamma_2}{(kn)^2\pi^2} \right] +$$

$$\frac{2(kn)^2\pi}{\gamma_1\gamma_2} \left\{ \frac{(-1)^s 3A - \left[1 - (-1)^s\right] B}{\gamma_2\pi} \right\} \left[1 - \frac{\cosh \gamma_1 \sinh \gamma_2}{\sinh \gamma_1 \cosh \gamma_2} \frac{\gamma_1\gamma_2}{(kn)^2\pi^2} \right]$$

$$P_2 = -2\left\{ \frac{3A + \left[1 - (-1)^s\right] B}{\gamma_2\pi} \right\} - \left\{ \sum_{m=1} \frac{m^3 A_m \left[(-1)^s + (-1)^m\right]}{m^2 + (kn)^2 + \lambda_1} - \frac{6A\left[1 + (-1)^s\right]}{\gamma_2{}^2\pi} \right\} \frac{\sinh \gamma_2}{\cosh \gamma_2}$$

$$P_3 = \frac{\gamma_1\gamma_2}{(kn)^2\pi^2} \left\{ \sum_{m=1}^{\infty} \frac{m^3 A_m \left[(-1)^s + (-1)^m\right]}{m^2 + (kn)^2 + \lambda_1} - \frac{6A\left[1 + (-1)^s\right]}{\gamma_2{}^2\pi} \right\} +$$

$$2\left\{ \frac{3A + \left[1 - (-1)^s\right] B}{\gamma_2\pi} \right\} \frac{\cosh \gamma_1}{\sinh \gamma_1} + 2\left\{ \frac{(-1)^s 3A - \left[1 - (-1)^s\right] B}{\gamma_2\pi} \right\} \frac{1}{\sinh \gamma_1}$$

When γ_1 and γ_2 are large, so that $\sinh \gamma_1 \approx \cosh \gamma_1$ and $\sinh \gamma_2 \approx \cosh \gamma_2$, equation (26) can be simplified to obtain:

$$H = \frac{1}{P'} \left[1 - \frac{\gamma_1\gamma_2}{(kn)^2\pi^2} \right] \left\{ \sum_{m=1}^{\infty} \frac{\left[(-1)^{m+s} + 1\right] m^3 A_m}{m^2 + (kn)^2 + \lambda_1} - \frac{6A\left[1 + (-1)^s\right]}{\gamma_2{}^2\pi} \right\} +$$

$$\frac{2(kn)^2\pi^2}{\gamma_1\gamma_2} \left\{ \frac{(-1)^s 3A - \left[1 - (-1)^s\right] B}{\gamma_2\pi} \right\} \tag{27}$$

where

$$P' = \gamma_2 \left\{ \left[\frac{(kn)^2\pi^2}{\gamma_1\gamma_2} - 1 \right] + \left[\frac{\gamma_1\gamma_2}{(kn)^2\pi^2} - 1 \right] \right\}$$

It can be seen that equation (25) represents two independent systems of linear, homogeneous, algebraic equations in A_s. While the system corresponding to the odd numbers $s = 2m - 1$ with $m = 1, 2, \ldots$ represents symmetrical buckling, the system corresponding to the even numbers $s = 2m$ governs antisymmetrical buckling.

NUMERICAL EVALUATION OF RESULTS OBTAINED

BY LEGGETT'S METHOD

The homogeneous set of linear equations derived in the preceding section was solved numerically for many different values of the geometric and mechanical quantities involved. As the minimum value of the buckling stress obtains when the buckling is symmetric, only odd values had to be attributed to s. The first approximation to the true value was calculated by assuming that the deflected shape could be represented by a polynomial and a single sine function of the coordinate ξ (or x) and by considering n a given constant. The transverse deflections are then given by

$$w = A_1\left(\sin \pi\xi + \pi\xi^2 - \pi\xi\right) \sin n\pi\eta \tag{28}$$

which obviously satisfies the boundary conditions. The set of equations represented by equation (25) consists of a single equation in this case, and the compressive stress corresponding to the nontrivial solution of this equation is the buckling stress.

In the second approximation two terms of the sine series were considered. Equations (8c), (15e), and (16) yielded the following expression for the deflected shape:

$$w = \left[A_1 \sin \pi\xi + A_3 \sin 3\pi\xi + \pi\left(A_1 + 3A_3\right)\left(\xi^2 - \xi\right)\right] \sin n\pi\eta \tag{29}$$

This equation again satisfies all the boundary conditions. Equation (25) yields two simultaneous equations in this case, one for $s = 1$, and the second for $s = 3$. The value of the compressive stress that makes the determinant of the two equations vanish is the buckling stress. It is well to remember, however, that equation (25) was obtained after all functions appearing in the solution were expanded in Fourier series. The first single term, or the first two terms, of these series do not satisfy the boundary conditions rigorously, and for this reason the critical stress calculated from the equations is only approximate. It is shown in appendix A that the value obtained in this process is always smaller than the true buckling stress.

It was found convenient to define the following nondimensional parameters:

$$\left.\begin{aligned} R &= G_c/F\sigma_{cr,o} \\ r &= c/t \\ F &= 1 + 3(1 + r)^2 \end{aligned}\right\} \tag{30}$$

If they are used, equations (24) can be written as

$$\left.\begin{aligned} \lambda_1 &= (2/3)rFR/(1 + r)^2 \\ \lambda_2 &= 2rFR \end{aligned}\right\} \tag{31}$$

The buckling stress of the sandwich plate was given in the form

$$\sigma_{cr} = CF\sigma_{cr,o} = (\lambda/4F)\sigma_{cr,o} \tag{32}$$

which equation implicitly defines the critical stress factor C.

In figure 4, by way of example, the values of C are plotted against the plate aspect ratio L_y/L_x for the fixed values $r = 39$ and $R = 0.3$. Different choices of n give different critical stresses and, as with homogeneous plates, the value of n yielding the smallest critical stress is the only one of practical importance. The curves obtained from the first and second approximations do not differ much and for this reason it was not considered necessary to calculate a third approximation. It can also be seen that for large values of L_y/L_x, say above 2, the aspect ratio has little influence upon the buckling stress. In this range it is permissible, therefore, to use the minimum value of C which, incidentally, is independent of n.

The minimum values of the critical stress factor are shown in figure 5. The abscissa is $r = c/t$ and the parameter of the family of curves is the ratio $R = G_c/F\sigma_{cr,o}$. As the values of this parameter range from 0 to ∞, and c/t ranges from 0 to 100, all possible sandwich panels are covered in the diagram.

In the limiting case of a homogeneous panel, that is, when the value of R approaches infinity, the critical stress factor becomes $6.98/4 = 1.745$.

The reduction of results obtained by the Leggett method to a thin homogeneous plate is given in appendix B. It was stated earlier that Leggett's method yields lower bounds for the buckling stress. A better estimate of the true values of the buckling stress can be had if upper bounds are also established. For this reason a different solution of the problem is given in the next section in which Galerkin's method is used in the calculations.

SOLUTION BY GALERKIN'S METHOD

In applying Galerkin's method to this problem (see appendix C), it is convenient to introduce the variable ζ by means of the following relation:

$$\zeta = 2\xi - 1$$

This means that the origin of coordinates is placed at the geometric center of the sandwich plate. The differential equations (12), (13), and (14) become

$$- \frac{4}{\pi^2} F_n''(\zeta) + \left[\frac{(1 - \mu)}{2} (kn)^2 + \lambda_1\right] F_n(\zeta) +$$

$$\frac{(1 + \mu)kn}{\pi} G_n'(\zeta) + \lambda_1 c_o H_n'(\zeta) = 0 \qquad (33)$$

$$- \frac{(1 + \mu)kn}{\pi} F_n'(\zeta) - \frac{2(1 - \mu)}{\pi^2} G_n''(\zeta) +$$

$$\left[(kn)^2 + \lambda_1\right] G_n(\zeta) + \frac{kn\pi}{2} \lambda_1 c_o H_n(\zeta) = 0 \tag{34}$$

$$- \frac{4\lambda_2}{c_o} F_n'(\zeta) + \frac{2\lambda_2}{c_o} kn\pi G_n''(\zeta) + \frac{16}{\pi^2} H_n''''(\zeta) - 4\left[2(kn)^2 + \lambda_2\right] H_n''(\zeta) +$$

$$(kn)^2\pi^2 \left[(kn)^2 + \lambda_2 - \lambda\right] H_n(\zeta) = 0 \tag{35}$$

and the boundary conditions become:

$$F_n(\zeta) = 0 \qquad \text{at} \quad \zeta = 1, -1 \tag{36}$$

$$G_n(\zeta) = 0 \qquad \text{at} \quad \zeta = 1, -1 \tag{37}$$

$$H_n(\zeta) = H_n'(\zeta) = 0 \qquad \text{at} \quad \zeta = 1, -1 \tag{38}$$

The solution of the differential equations can be assumed as:

$$F_n(\zeta) = \sum_{m=1,2,\ldots}^{\infty} A_m \sin m\pi\zeta \tag{39}$$

$$G_n(\zeta) = \sum_{m=1,2,\ldots}^{\infty} B_m \cos \left(\frac{2m - 1}{2}\right)\pi\zeta \tag{40}$$

$$H_n(\zeta) = \sum_{m=1,3,\ldots}^{\infty} C_m \varphi_m(\zeta) \tag{41}$$

Each individual term of the series satisfies all the required boundary conditions provided the functions $\varphi_m(\zeta)$ are chosen as the normal modes of vibration of a beam clamped at both ends (see appendix D). Because of the orthogonality of the trigonometric and the normal functions φ (see equations (D5), (D6), and (D7) of appendix D) the symmetrical and antisymmetrical buckling modes can be considered separately. Since the lowest mode of buckling is symmetrical, $H_n(\zeta)$ and $G_n(\zeta)$ were assumed symmetrical and $F_n(\zeta)$ was assumed antisymmetrical about the origin.

In the actual calculations the entire infinite series given in equations (39) and (40) will be considered, but only a finite number of the φ functions will be taken into account.

Insertion of $F_n(\zeta)$, $G_n(\zeta)$, and $H_n(\zeta)$ into equations (33), (34), and (35) leads to

$$\sum_{m=1,2,\ldots}^{\infty} \left[\left(4m^2 + a_1\right)A_m \sin m\pi\zeta - a_2\left(\frac{2m - 1}{2}\right)B_n \sin\left(\frac{2m - 1}{2}\pi\zeta\right)\right] +$$

$$a_3 \sum_{s=1,3,\ldots}^{r} C_s\varphi_s'(\zeta) = e_1(\zeta) \tag{33a}$$

$$\sum_{m=1,2,\ldots}^{\infty} \left\{-b_1 mA_m \cos m\pi\zeta + \left[b_2\left(\frac{2m - 1}{2}\right)^2 + b_3\right]B_m \cos\left(\frac{2m - 1}{2}\pi\zeta\right)\right\} +$$

$$b_4 \sum_{s=1,3,\ldots}^{r} C_s\varphi_s(\zeta) = e_2(\zeta) \tag{34a}$$

$$\sum_{m=1,2}^{\infty} \left[-c_1(m\pi)A_m \cos m\pi\zeta + c_2 B_m \cos\left(\frac{2m - 1}{2}\pi\zeta\right)\right] +$$

$$\sum_{s=1,3}^{r} \left(c_3 k_s^{\,4} + c_5\right)C_s\varphi_s(\zeta) - c_4 \sum_{s=1,3,\ldots}^{r} C_s\varphi_s''(\zeta) = e_3(\zeta) \tag{35a}$$

where

$$a_1 = \frac{(1 - \mu)}{2}(kn)^2 + \lambda_1 \qquad a_2 = (1 + \mu)kn \qquad a_3 = \lambda_1 c_o$$

$$b_1 = (1 + \mu)kn \qquad b_2 = 2(1 - \mu) \qquad b_3 = (kn)^2 + \lambda_1$$

$$b_4 = (kn)\pi\lambda_1 c_o/2$$

$$c_1 = 4\lambda_2/c_o \qquad c_2 = 2\lambda_2 kn\pi/c_o \qquad c_3 = 16/\pi^2 \qquad c_4 = 4\left[2(kn)^2 + \lambda_2\right]$$

$$c_5 = (kn)^2\pi^2\left[(kn)^2 + \lambda_2 - \lambda\right]$$

If the following integrals are formed (see appendix C),

$$I_1 = \int_{-1}^{1} e_1(\zeta) \sin (t\pi\zeta)\, d\zeta \qquad t = 1, 2, 3, \ldots$$

$$I_2 = \int_{-1}^{1} e_2(\zeta) \cos \left(\frac{2t - 1}{2}\pi\zeta\right) d\zeta \qquad t = 1, 2, 3, \ldots$$

$$I_3 = \int_{-1}^{1} e_3(\zeta)\varphi_q(\zeta)\, d\zeta \qquad q = 1, 3, 5, \ldots$$

Galerkin's equations can be obtained by setting the triply infinite set of integrals equal to zero:

$$I_1 = I_2 = I_3 = 0$$

Consequently

$$\left(4t^2 + a_1\right)A_t + (-1)^t \frac{a_2 t}{\pi} \sum_{m=1}^{\infty} \left[\frac{(-1)^m(2m - 1)}{\left(\frac{2m - 1}{2}\right)^2 - t^2} B_m\right] - a_3 t\pi \sum_{s=1}^{r} C_s \alpha_{st} = 0 \tag{42}$$

$$\frac{(-1)^t b_1(2t - 1)}{\pi} \sum_{m=1}^{\infty} \left[\frac{(-1)^m m}{\left(\frac{2t - 1}{2}\right)^2 - m^2} A_m\right] +$$

$$\left[b_2\left(\frac{2t - 1}{2}\right)^2 + b_3\right]B_t + b_4 \sum_{s=1}^{r} C_s \beta_{st} = 0 \tag{43}$$

$$\sum_{m=1}^{\infty} \left[-c_1(m\pi)\alpha_{qm}A_m + c_2\beta_{qm}B_m\right] + \left(c_3 k_q{}^4 + c_5\right)C_q - c_4 \sum_{s=1}^{r} C_s \gamma_{sq} = 0 \tag{44}$$

In these equations m and t take on the values 1, 2, 3, . . ., and s and q the values 1, 3, 5, ... The parameters α, β, and γ are defined as

$$\alpha_{st} = \int_{-1}^{1} \varphi_s(\zeta) \cos t\pi\zeta \, d\zeta$$

$$= -\frac{1}{t\pi} \int_{-1}^{1} \varphi_s{}'(\zeta) \sin t\pi\zeta \, d\zeta$$

$$= \frac{(-1)^t 2\varphi_s{}'''(1)}{k_s{}^4 - (t\pi)^4}$$

$$\beta_{st} = \int_{-1}^{1} \varphi_s(\zeta) \cos \left(\frac{2t - 1}{2}\right)\pi\zeta \, d\zeta$$

$$= \frac{(-1)^{t+1} (2t - 1)\pi\varphi_s''(1)}{k_s^4 - \left(\frac{2t - 1}{2}\right)^4 \pi^4}$$

$$\gamma_{st} = \int_{-1}^{1} \varphi_s''(\zeta)\varphi_t(\zeta) \, d\zeta$$

The numerical values of γ_{st} needed for calculation are listed in appendix D.

Equations (42), (43), and (44) form a system of linear homogeneous algebraic equations, and the vanishing of their determinant yields the condition for the evaluation of the critical load. It can be shown that the infinite series given in equations (39) to (41) represent a rigorous solution of the problem if the coefficients of the terms are calculated from the infinite set of equations (42) to (44) with $r \longrightarrow \infty$. Moreover, a finite number of terms gives just as good an approximate solution as the Rayleigh-Ritz method employing the same terms. However, the work involved is large.

For this reason the determinant was not evaluated but the coefficients A_m and B_m were expressed in terms of the coefficients C_m from equations (42) and (43), and then substituted in equation (44). This procedure leads to an infinite set of homogeneous equations linear in the C_m. The summations indicated in the equations were carried out with the aid of complex integration as shown in appendix E.

Equation (42) was solved first for A_m:

$$A_m = \frac{(-1)^{m+1} m\pi}{4m^2 + a_1}\left[\frac{a_2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n (2n - 1)}{\left(\frac{2n - 1}{2}\right)^2 - m^2} B_n - 2a_4 \sum_{s=1}^{r} \frac{C_s \varphi_s''(1)}{k_s^4 - (m\pi)^4}\right] \tag{45}$$

When this value is inserted into equation (43) and the summation signs are interchanged,[1] the following equation is obtained:

$$\left(a_2/\pi^2\right)\sum_{n=1}^{\infty}(-1)^n(2n-1)B_n\left\{\sum_{m=1}^{\infty}\frac{m^2}{\left(4m^2+a_1\right)\left[\left(\frac{2t-1}{2}\right)^2-m^2\right]\left[\left(\frac{2n-1}{2}\right)^2-m^2\right]}\right\} -$$

$$2a_3\sum_{s=1}^{r}C_s\varphi_s'''(1)\left\{\sum_{m=1}^{\infty}\frac{m^2}{\left(4m^2+a_1\right)\left[\left(\frac{2t-1}{2}\right)^2-m^2\right]\left[k_s^4-(m\pi)^4\right]}\right\} -$$

$$\frac{(-1)^t\left[b_2\left(\frac{2t-1}{2}\right)^2+b_3\right]B_t}{b_1(2t-1)}+\frac{b_4\pi}{b_1}\sum_{s=1}^{r}\frac{C_s\varphi_s''(1)}{k_s^4-\left(\frac{2t-1}{2}\right)^4\pi^4} = 0 \tag{46}$$

With the notation

$$a_o^2 = a_1/4 \qquad t_o = (2t-1)/2 \qquad n_o = (2n-1)/2 \qquad k_{so} = k_s/\pi$$

and utilizing the summation by complex integration developed in appendix E one obtains:

$$\sum_{m=1}^{\infty}\frac{m^2}{\left(4m^2+a_1\right)\left[\left(\frac{2t-1}{2}\right)^2-m^2\right]\left[\left(\frac{2n-1}{2}\right)^2-m^2\right]} = \frac{1}{4}\sum_{m=1}^{\infty}\frac{m^2}{\left(m^2+a_o^2\right)\left(m^2-t_o^2\right)\left(m^2-n_o^2\right)}$$

$$= -\frac{1}{8}\frac{\pi a_o \coth \pi a_o}{\left(a_o^2+t_o^2\right)\left(a_o^2+n_o^2\right)}$$

when $t_o \neq n_o$

$$= -\frac{1}{8}\frac{\pi a_o \coth \pi a_o}{\left(a_o^2+t_o^2\right)^2}+\frac{\pi^2}{16\left(a_o^2+t_o^2\right)}$$

when $t_o = n_o$

[1]This can be justified because of the absolute convergence of these series.

and

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(4m^2 + a_1\right)\left[\left(\frac{2t - 1}{2}\right)^2 - m^2\right]\left[k_s^4 - (m\pi)^4\right]} = \frac{1}{4\pi^4} \sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a_o^2\right)\left(m^2 - t_o^2\right)\left(m^4 - k_{so}^4\right)}$$

$$= \frac{1}{\left(8\pi^3\right)\left(a_o^4 - k_{so}^4\right)} \left[\frac{a_o \coth \pi a_o}{\left(a_o^2 + t_o^2\right)} + \frac{\cot \pi k_{so}\left(a_o^2 t_o^2 - k_{so}^4\right)}{k_{so}\left(t_o^4 - k_{so}^4\right)}\right]$$

If these equations as well as equation (D9) of appendix D are taken into account, equation (46) becomes:

$$\frac{a_2}{\pi} \sum_{n=1}^{\infty} (-1)^n n_o B_n \frac{a_o \coth \pi a_o}{\left(a_o^2 + n_o^2\right)} + (-1)^t B_t \frac{4\left[(1 - \mu)t_o^2 + 2a_o^2\right]\left(t_o + d_o^2\right)}{t_o(1 + \mu)kn} +$$

$$\frac{a_3}{\pi^2} \sum_{s=1}^{r} C_s \varphi_s''(1) \left\{\frac{a_o k_{so} \coth \pi a_o \tanh \pi k_{so}}{\left(a_o^4 - k_{so}^4\right)} +\right.$$

$$\left.\frac{\left(a_o + t_o^2\right)}{\left(t_o^4 - k_{so}^4\right)}\left[\frac{(1 - \mu)}{(1 + \mu)} + \frac{a_o^2\left(a_o^2 - t_o^2\right)}{a_o^4 - k_{so}^4}\right]\right\} = 0 \qquad (47)$$

where $d_o^2 = b_3/4$.

Similarly, if the value of A_m as given in equation (45) is inserted into equation (44), and the series are summed (see equations (E7), (E8), and (E9) of appendix E) the result is:

$$\frac{a_2 c_1 \varphi_q'''(1)}{2\pi^3} \frac{a_o \coth \pi a_o}{a_o^4 - k_o^4} \sum_{n=1}^{\infty} \frac{(-1)^n n_o B_n}{\left(a_o^2 + n_o^2\right)} +$$

$$\frac{c_2 \varphi_q''(1)}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n n_o B_n}{\left(n_o^4 - k_o^4\right)} \left[(1 - \mu) + \frac{(1 + \mu) a_o^2 \left(a_o^2 - n_o^2\right)}{\left(a_o^4 - k_o^4\right)}\right] +$$

$$\frac{c_1 a_3}{2\pi^5} \varphi_q'''(1)\left(X_{qq} + X_{sq}\right) + \left(c_3 k_q^4 + c_5\right) C_q - c_4 \sum_{s=1}^{r} C_s \gamma_{sq} = 0 \qquad (48)$$

where

$$X_{qq} = C_q \varphi_q'''(1) \left[\frac{a_o \coth \pi a_o}{\left(a_o^4 - k_{qo}^4\right)^2} + \frac{a_o^2 \coth \pi k_{qo} \left(a_o^4 - 5k_o^4\right)}{4k_{qo}^5 \left(a_o^4 - k_{qo}^4\right)^2} + \right.$$

$$\left. \frac{\pi}{4k_{qo}^2 \left(a_o^4 - k_{qo}^4\right) \sin^2 \pi k_{qo}} - \frac{\pi}{4k_{qo}^4 \left(a_o^2 - k_{qo}^2\right)}\right]$$

when $s = q$

and

$$X_{sq} = \sum_{s=1}^{r} C_s \varphi_s'''(1) \left\{\frac{a_o \coth \pi a_o}{\left(a_o^4 - k_{qo}^4\right)\left(a_o^4 - k_{so}^4\right)} + \right.$$

$$\left. \frac{a_o^2}{\left(k_{qo}^4 - k_{so}^4\right)} \left[\frac{\coth \pi k_{so}}{k_{so}\left(a_o^4 - k_{so}^4\right)} - \frac{\coth \pi k_{qo}}{k_{qo}\left(a_o^4 - k_{qo}^4\right)}\right]\right\}$$

when $s \neq q$.

If $D_n = B_n/3$ and the values of a_2 and a_3 as previously defined are inserted into equation (47), it becomes:

$$\frac{(1+\mu)kn}{\pi^3}\sum_{n=1}^{\infty}(-1)^n n_o D_n \frac{a_o \coth \pi a_o}{\left(a_o^2 + n_o^2\right)} + (-1)^t D_t \frac{4\left[(1-\mu)t_o^2 + 2a_o^2\right]\left(t_o^2 + d_o^2\right)}{t_o(1+\mu)(kn)\pi^2} +$$

$$\frac{1}{\pi^4}\sum_{s=1}^{r} C_s\varphi_s''(1)\left\{\frac{a_o k_{so} \coth \pi a_o \tanh \pi k_{so}}{\left(a_o^4 - k_{so}^4\right)} + \frac{\left(a_o^2 + t_o^2\right)}{\left(t_o^4 - k_{so}^4\right)}\left[\frac{(1-\mu)}{(1+\mu)} +\right.\right.$$

$$\left.\left.\frac{a_o^2\left(a_o^2 - t_o^2\right)}{\left(a_o^4 - k_{so}^4\right)}\right]\right\} = 0 \tag{49}$$

By means of equations (48) and (49) it is possible to express any D_t in terms of D_p and a linear combination of C_s as follows:

$$(-1)^t D_t \frac{\left(t_o^2 + b_o^2\right)\left(t_o^2 + d_o^2\right)}{t_o} = (-1)^p D_p \frac{\left(p_o^2 + b_o^2\right)\left(p_o^2 + d_o^2\right)}{p_o} -$$

$$\frac{kn}{4\pi^2}\sum_{s=1}^{r} C_s\varphi_s''(1)\left[\frac{\left(t_o^2 + b_o^2\right)}{\left(t_o^4 - k_{so}^4\right)} - \frac{\left(p_o^2 + b_o^2\right)}{\left(p_o^4 - k_{so}^4\right)}\right]$$

where $b_o^2 = a_o/\left[2(1 - \mu)\right]$.

It can be seen from the above equations that D_p is of the order of magnitude of $1/p^3$, and p can be chosen very large so that $\left(p_o^2 + b_o^2\right)/\left(p_o^4 - k_{so}^4\right)$ approaches zero, since both b_o and k_{so} are finite. Hence the above equation may be given as:

$$(-1)^t Dt = \frac{t_o}{\left(t_o^2 + b_o^2\right)\left(t_o^2 + d_o^2\right)}\left[(-1)^p K_p D_p - \frac{kn\left(t_o^2 + b_o^2\right)}{4\pi^2}\sum_{s=1}^{r}\frac{C_s\varphi_s''(1)}{\left(t_o^4 - k_{so}^4\right)}\right] \tag{50}$$

where

$$K_p = \frac{\left(p_o^2 + b_o^2\right)\left(p_o^2 + d_o^2\right)}{p_o}$$

Now substitution of D_t from equation (50) into equation (49), with $t = p$, makes it possible to express D_p in terms of linear combinations of the C_s terms. Because of equations (E13) and (E15) of appendix E and after further simplification the final equation is:

$$KD_p = \frac{1}{\pi^4}\frac{\lambda_1}{\left[4d_o \tanh \pi d_o - \frac{(kn)^2}{b_o}\tanh \pi b_o\right]}\sum_{s=1}^{r} C_s\varphi_s''(1)\left\{\frac{1}{\left(d_o^4 - k_{so}^4\right)}\times\left[d_o \tanh \pi d_o + \frac{(1 + \mu)(kn)^2\left(a_o^2 + d_o^2\right)k_{so}\tanh \pi k_{so}}{8\left(a_o^4 - k_{so}^4\right)}\right] - \frac{k_{so}\tanh \pi k_{so}}{\left(a_o^4 - k_{so}^4\right)}\right\} \tag{51}$$

where

$$K = (-1)^p \frac{4(1 - \mu)K_p}{(1 + \mu)(kn)\pi^2}$$

In equation (51), if $t = p$ is chosen a very large number, the last term approaches the value $a_o^2/(a_o^4 - k_{so}^4)$. After substitution of the values of a_2, a_3, and c_1 into equation (48), multiplication by $(a_o^4 - k_o^4)/\left[2\lambda_1\lambda_2\varphi'''(1)\right]$, and subtraction of equation (49), with $t = p$, from it the following expression is obtained:

$$- \frac{4(1 - \mu)K_p}{(1 + \mu)(kn)\pi^2}(-1)^p D_p + \frac{kn}{\pi^3 k_{qo} \tanh \pi k_{qo}} \sum_{n=1}^{\infty} \frac{(-1)^n n_o D_n}{\left(n_o^4 - k_{qo}^4\right)} \times$$

$$\left[2a_o^4 - (1 - \mu)k_{to}^4 - (1 + \mu)a_o^2 n_o^2\right] - \frac{1}{\pi^4}\left(Y_{qq} + Y_{sq}\right) +$$

$$\frac{a_o^4 - k_{qo}^4}{2\lambda_1\lambda_2\varphi'''(1)}\left(c_3 k_q + c_5\right)C_q - c_4 \sum_{r=1}^{\infty} C_s \gamma_{sq} = 0 \qquad (52)$$

where

$$Y_{qq} = C_q \varphi_q''(1)\left[\frac{\left(a_o^2 + k_{qo}^2\right)\pi k_{qo} \tanh \pi k_{qo} - a_o^2}{4k_{qo}^4} + \frac{\pi}{2k_{qo} \sin 2\pi k_{qo}}\right]$$

when $s = q$ and

$$Y_{sq} = \sum_{s=1}^{r} C_s \varphi_s''(1)\left\{\frac{a_o^2}{\left(k_{so}^4 - k_{qo}^4\right)}\left[1 - \frac{\left(k_{so} \tanh \pi k_{so}\right)}{\left(k_{qo} \tanh \pi k_{qo}\right)}\right]\right\}$$

when $s \neq q$.

Finally through substitution of D_n from equation (51) into the second term of equation (52), use of relations (E21) to (E24), and summing up the series (see equations (E14), (E16), and (E20) of appendix E), the following system of equations is obtained:

$$P_q \sum_{s=1}^{r} C_s \varphi_s''(1) \left\{ \frac{1}{\left(d_o^4 - k_{so}^4\right)} \left[d_o \tanh \pi d_o + \right.\right.$$

$$\left.\left. \frac{(1 + \mu)(kn)^2\left(a_o^2 + d_o^2\right)k_{so} \tanh \pi k_{so}}{\left(a_o^4 - k_{so}^4\right)} \right] - \frac{k_{so} \tanh \pi k_{so}}{\left(a_o^4 - k_{so}^4\right)} \right\} -$$

$$\sum_{s=1}^{r} C_s \varphi_s''(1) \left[\Phi_{sq} + \frac{(1 - \mu)(kn)^2}{8k_{qo} \tanh \pi k_{qo}} \Psi_{sq} \right] +$$

$$\frac{\pi^4\left(a_o^4 - k_{qo}^4\right)}{2\lambda_1\lambda_2\varphi_q'''(1)} \left\{ 16k_{qo}^4 + (kn)^2\left[(kn)^2 + \lambda_2 - \lambda\right]\pi^2 C_q - \right.$$

$$\left. 4\left[2(kn)^2 + \lambda_2\right] \right\} \sum_{s=1}^{r} C_s \gamma_{sq} = 0 \tag{53}$$

for $q = 1, 3, 5, \ldots r$. In these equations

$$P_q = \frac{\lambda_1}{\left[4d_o \tanh \pi d_o - \frac{(kn)^2}{b_o} \tanh \pi b_o\right]} \left\{ \frac{\left(a_o^4 - k_{qo}^4\right)}{\left(d_o^4 - k_{qo}^4\right)} + \right.$$

$$\left. \frac{d_o^2 - a_o^2}{k_{qo} \tanh \pi k_{qo}} \left[\frac{d_o\left(d_o^2 + a_o^2\right)}{\left(d_o^4 - k_{qo}^4\right)} \tanh \pi d_o - \frac{b_o \tanh \pi b_o - d_o \tanh \pi d_o}{\left(b_o^2 - d_o^2\right)} \right] \right\}$$

$$\Phi_{sq} = \frac{\left(a_o^2 + k_{qo}^2\right)\pi k_{qo} \tanh \pi k_{qo} - a_o^2}{4k_{qo}^4} + \frac{\pi}{2k_{qo} \sin 2\pi k_{qo}} \qquad \text{when} \quad s = q$$

$$\Phi_{sq} = \frac{a_o^2}{\left(k_{so}^4 - k_{qo}^4\right)}\left(1 - \frac{k_{so} \tanh \pi k_{so}}{k_{qo} \tanh \pi k_{qo}}\right) \qquad \text{when} \quad s \neq q$$

$$\Psi_{sq} = \frac{\pi\left(a_o^2 + k_{qo}^2\right)\left(b_o^2 - k_{qo}^2\right)}{4k_{qo}^4\left(d_o^2 - k_{qo}^2\right)} - \left[1 + \frac{\left(b_o^2 - d_o^2\right)\left(a_o^2 + d_o^2\right)}{\left(d_o^4 - k_{qo}^4\right)}\right]\left[\frac{\pi \sec^2 \pi k_{qo}}{4k_{qo}^2} - \right.$$

$$\left.\frac{\tanh \pi k_{qo}\left(d_{qo} + 3k_{qo}^4\right)}{4k_{qo}^3\left(d_o^4 - k_{qo}^4\right)} + \frac{d_o \tanh \pi d_o}{d_o^4 - k_{qo}^4}\right] \qquad \text{when} \quad s = q$$

$$\Psi_{sq} = \left[1 + \frac{(1 + \mu)\lambda}{4(1 - \mu)} \frac{\left(a_o^2 + d_o^2\right)}{\left(d_o^4 - k_{qo}^4\right)}\right]\left[\frac{-d_o \tanh \pi d_o}{\left(d_o^4 - k_{so}^4\right)} + \frac{k_{qo} \tanh \pi k_{qo}}{\left(k_{qo}^4 - k_{so}^4\right)} + \right.$$

$$\left.\frac{\left(d_o^4 - k_{qo}^4\right)}{\left(d_o^4 - k_{so}^4\right)} \frac{k_{so} \tanh \pi k_{so}}{\left(k_{so}^4 - k_{qo}^4\right)}\right] \qquad \text{when} \quad s \neq q$$

Equation (53) represents a system of homogeneous linear algebraic equations in the constants C_s. By setting the rth-order determinant equal to zero, successive solutions for the critical load for $r = 1, 3, \ldots$ may be obtained.

NUMERICAL EVALUATION OF RESULTS OBTAINED

BY GALERKIN'S METHOD

As a first approximation, equation (53) with $q = s = 1$ and with appropriate numerical values of the mode shape constants obtained from appendix D may be given as

$$-P_1\left\{\frac{1}{\left(d_o^4 - k_{10}^4\right)}\left[d_o \tanh \pi d_o + 0.1201910888\,\frac{(kn)^2\left(a_o^2 + d_o^2\right)}{\left(a_o^4 - k_{10}^4\right)}\right] - \right.$$

$$\left.\frac{0.73963747}{\left(a_o^4 - k_{10}^4\right)}\right\} - 0.1183012(kn)^2\left\{2.44539946\,\frac{\left(a_o^2 + k_{10}^2\right)\left(b_o^2 - k_{10}^2\right)}{\left(d_o^2 - k_{10}^2\right)} - \right.$$

$$1 + \frac{0.4642857\lambda_1\left(a_o^2 + d_o^2\right)}{\left(d_o^4 - k_{10}^4\right)}\left[2.7236504 - 0.575730\,\frac{\left(d_o^4 + 3k_{10}^4\right)}{\left(d_o^4 - k_{10}^4\right)} + \right.$$

$$\left.\left.\frac{d_o \tanh \pi d_o}{\left(d_o^4 - k_{10}^4\right)}\right]\right\} - \left(1.030314267a_o^2 - 1.06186907\right) +$$

$$\frac{3.306090015(kn)^2\left(a_o^4 - k_{10}^4\right)}{\lambda_1\lambda_2}\left\{\left[\frac{5.1387804}{(kn)^2} + (kn)^2 + 2.304656\right] + \right.$$

$$\left.\lambda_2\left[1 + \frac{1.2464478}{(kn)^2}\right] - \lambda\right\} = 0 \tag{54}$$

where

$$P_1 = \frac{\lambda_1}{4d_o \tanh \pi d_o - \left(1/b_o\right)(kn)^2 \tanh \pi b_o} \left\{\frac{\left(a_o^4 - k_{10}^4\right)}{\left(d_o^4 - k_{10}^4\right)} + \right.$$

$$0.21970224(kn)^2\left[\frac{d_o\left(a_o^2 + d_o^2\right) \tanh \pi d_o}{\left(d_o^4 - k_{10}^4\right)} - \right.$$

$$\left.\left.\frac{2.153846\left(b_o \tanh \pi b_o - d_o \tanh \pi d_o\right)}{\lambda_1}\right]\right\}$$

and

$$a_o^2 = (1/4)\left[(1 - \mu)(1/2)(kn)^2 + \lambda_1\right]$$

$$b_o^2 = 2a_o^2 / (1 - \mu)$$

$$d_o^2 = \left[(kn)^2 + \lambda_1\right] / 4$$

$$k_{10} = k_1 / \pi = 0.752809375$$

Calculations using the above equation were made for the parameters $r = 39$, $R = 0.3$, and $n = 1$ and for various values of the aspect ratio L_y/L_x; the results are shown as the uppermost curve in figure 6. This solution gives the minimum value of the buckling stress factor C as 1.430. For a second approximation, two equations of the type of equation (52) were used with q taken as 1 and 3 and s as 1 and 3. The results of the calculations were plotted in the curve labeled "upper bound." The minimum value of C for this case was found to be 1.405. It is noted that both these values are higher than the first and second approximations obtained by the Leggett method which were also plotted in figure 6 for comparison purposes. The values obtained by the Leggett method approach the true values from below while values obtained from the Galerkin method approach them from above.

The percentage difference between the minimums of the two first approximations (when referred to the lower value) is 11 percent, and that between the two second approximations is 7 percent. For practical calculations the arithmetic mean of the two second approximations can be taken as the true value of the critical stress factor. In the case just discussed the mean is 1.352 and the error is less than 3.5 percent when this mean is used.

Table I shows a few sample values of lower and upper bounds and also their arithmetic means which will be called the true values of the minimum critical stress factor C_{min}. All the values calculated are presented in figure 5. Figure 7 is the plot of the true values which are recommended for use in practical calculations.

NUMERICAL EXAMPLES

As a first example, the buckling load of an aluminum cellular cellulose-acetate sandwich panel is calculated with the aid of figure 7. The panel was tested and reported as panel (1-1) in table 11 of reference 4. In the notation of the present report, the data of the panel are:

$$t = 0.013 \text{ inch} \qquad c = 0.247 \text{ inch}$$

$$L_x = 39.82 \text{ inches} \qquad E = 9.9 \times 10^6 \text{ psi}$$

$$\mu = 0.3 \qquad G_c = 3500 \text{ psi}$$

The following parameters are calculated:

$$r = c/t = 19$$

$$F = 1 + 3(1 + r)^2 = 1201$$

$$\sigma_{cr,o} = \pi^2 E t^2 \Big/ \left[3\left(1 - \mu^2\right) L_x{}^2\right] = 3.81 \text{ psi}$$

$$R = G_c \Big/ \left(F \sigma_{cr,o}\right) = 0.764$$

From figure 7, the value C_{min} is approximately 1.32. Hence according to equation (30) the buckling stress is:

$$\left(\sigma_{cr}\right)_{min} = C_{min}F\sigma_{cr,o} = 1.32(1201)3.81 = 6040 \text{ psi}$$

and the buckling load is

$$Y_{min} = 2(0.013)6040 = 157.0 \text{ pounds per inch}$$

The test results (with $L_y = 33.00$ in.) of the Forest Products Laboratory give buckling loads ranging from 146 to 176 pounds per inch.

As a second example a panel with a balsa core (panel 1-1, table 10, reference 4) is considered. The data are:

$t = 0.012$ inch $\qquad c = 0.255$ inch

$L_x = 39.95$ inches $\qquad E = 9.9 \times 10^6$ psi

$\mu = 0.3$ $\qquad G_c = 19,000$ psi

The following parameters are calculated:

$$r = 21.25$$

$$F = 1486$$

$$\sigma_{cr,o} = 3.23 \text{ psi}$$

$$R = 3.96$$

From figure 7, the value of C_{min} is approximately 1.56. Hence the buckling stress is

$$\left(\sigma_{cr}\right)_{min} = 7486 \text{ psi}$$

and the buckling load is

$$Y_{min} = 179.7 \text{ pounds per inch}$$

The test results (with $L_y = 33.02$ in.) as obtained by the Forest Products Laboratory range from 164 to 176 pounds per inch. Hence the agreement between the theoretical results of the present investigation and the results of experimental tests carried out at the Forest Products Laboratory is satisfactory.

CONCLUDING REMARKS

The differential equations developed in TN 2225 have been solved for the buckling load of rectangular sandwich panels subjected to edgewise compression. Two solutions were obtained, one by the Leggett and the other by the Galerkin method. The former gave a lower and the latter an upper bound for the critical stress. In this manner the true buckling stress could be estimated fairly accurately as the arithmetic mean of the two bounds. The mean values were plotted for the entire practical range of the geometric and physical constants involved.

A comparison of results given in TN 2225 with results given in this report indicates that the difference in buckling stress in the region where the sandwich stiffness parameter is less than 0.025 and the ratio of core thickness to face thickness is less than 30 is negligible. These values characterize plates with a very weak core. For values of either the sandwich stiffness parameter or the ratio of core thickness to face thickness approaching infinity, the value of the minimum critical stress factor approaches unity in the simply supported case and 1.745 in the present problem. These values agree with those derived for ordinary isotropic thin plates.

Several numerical examples calculated from the theory were found to be in good agreement with results of tests carried out at the Forest Products Laboratory.

Polytechnic Institute of Brooklyn
Brooklyn, N. Y., April 20, 1950

APPENDIX A

BOUNDARY CONDITIONS

In order to ascertain whether successive solutions obtained by Leggett's method approach the buckling stress from above or below, the boundary conditions must be examined in some detail.

It may be recalled that $F_n(\xi)$, $G_n(\xi)$, and $H_n(\xi)$ as given by equations (23a), (23b), and (15e) satisfy all boundary conditions stated in equations (9), (10), and (11). When these functions are expanded into sine series, the boundary conditions of equations (9) and (10) and the first two of equation (11) are automatically satisfied, but whether the remaining two boundary conditions

$$H_n'(\xi) = 0 \quad \text{at} \quad \xi = 0,1 \tag{A1}$$

are satisfied is not obvious. After expanding into sine series, the first q terms of $H_n(\xi)$ may be expressed as:

$$H_n(\xi) = \sum_{m=1}^{q} A_m \sin m\pi\xi + \sum_{m=1}^{q} \left[4/(m\pi)^3\right] \left\{(-1)^m 3A - \left[1 - (-1)^m\right] B\right\} \sin m\pi\xi$$

and

$$H_n'(\xi) = \sum_{m=1}^{q} m\pi A_m \cos m\pi\xi + \sum_{m=1}^{q} \left[4/(m\pi)^2\right] \left\{(-1)^m 3A - \left[1 - (-1)^m\right] B\right\} \cos m\pi\xi$$

Hence from conditions (A1), when $\xi = 0,1$:

$$\sum_{m=1}^{q} m\pi A_m + \sum_{m=1}^{q} \left[4/(m\pi)^2\right] \left\{(-1)^m 3A - \left[1 - (-1)^m\right] B\right\} = 0$$

and

$$\sum_{m=1}^{q} (-1)m\pi A_m + \sum_{m=1}^{q} \left[4/(m\pi)^2\right] \left\{3A + \left[1 - (-1)^m\right]B\right\} = 0$$

Addition and subtraction of the above equations yield:

$$\sum_{m=1}^{q} m\pi A_m \left[1 + (-1)^m\right] + \sum_{m=1}^{q} \left[4/(m\pi)^2\right] \left[1 + (-1)^m\right] 3A = 0$$

$$\sum_{m=1}^{q} m\pi A \left[1 + (-1)^m\right] + \sum_{m=1}^{q} \left[4/(m\pi)^2\right] \left\{3A\left[(-1)^m - 1\right] - 2\left[1 - (-1)^m\right]B\right\} = 0$$

Insertion of the values of A and B from equations (16a) and (16b) gives after simplification:

$$\left(\sum_{k=1}^{q} kA_{2k}\right)\left[\pi^2/6 - \sum_{p=1}^{q} \left(1/p^2\right)\right] = 0 \qquad \text{(A2)}$$

$$\left[\sum_{k=1}^{q} (2k + 1)A_{2k+1}\right]\left\{\pi^2/8 - \sum_{p=1}^{q} \left[1/(2p + 1)\right]\right\} = 0 \qquad \text{(A3)}$$

If both $\sum_{k=1}^{q} kA_{2k}$ and $\sum_{k=0}^{q} (2k + 1)A_{2k+1}$ vanish simultaneously, the coefficients A, B, and C in equations (16a) to (16c) vanish and thus the boundary conditions are not satisfied. If one of the two sums does not vanish, the left-hand member of one of equations (A2) and (A3) must vanish. However, page 181 of reference 5 shows that:

$$\pi^2/6 = \sum_{p=1}^{\infty} \left(1/p^2\right) \qquad \text{(A4)}$$

$$\pi^2/8 = \sum_{p=1}^{\infty} \left[1/(2p + 1)\right]^2 \tag{A5}$$

Consequently equations (A2) and (A3) are satisfied only if an infinite number of terms in the series expansion $H_n''''(\xi)$ are taken into account. Hence the boundary conditions (A1) are not satisfied by approximate solutions which use a finite number of harmonics. It follows that the boundary conditions are relaxed and the critical stresses approach the true values from below (see reference 6).

APPENDIX B

REDUCTION OF RESULTS OBTAINED BY LEGGETT METHOD TO A THIN HOMOGENOUS PLATE

The sandwich plate is transformed into a homogeneous and isotropic thin plate if $c \longrightarrow 0$. When this is the case, λ_1 and λ_2 as defined in equations (24) vanish. Consequently equation (25) is simplified and in the case of symmetric buckling it can be written in the form

$$A_s s^4 + (kn)^2\left[(kn)^2 + 2s^2 - \lambda\right]\left[A_s - \frac{1}{s^2}(0.8105693)\left(A_1 + A_3 + A_5 + \ldots\right)\right] = 0 \tag{B1}$$

where $s = 1, 3, 5, \ldots$.

Values of the coefficient $\lambda = 16C$ in the formula for the buckling stress as obtained from equation (B1) are plotted in figure 8. The figure also shows Timoshenko's solution taken from page 320 of reference 7. It is noted that the third approximation gives a minimum value of $\lambda = 6.94$ which is only about 0.6 percent lower than Timoshenko's value of 6.98.

The system of equations corresponding to unsymmetrical buckling gives a higher buckling load for a plate of aspect ratio greater than 1 than that obtained from equation (B1). However, the equations for unsymmetrical buckling can be used conveniently to calculate the buckling stress of a thin plate clamped at one unloaded edge and simply supported at the other unloaded edge. With the notation

$$\lambda' = \sigma_{cr}\Big/\left[\pi^2Et^2/12\left(1 - \mu^2\right)(Lx/2)^2\right] = 4\sigma_{cr}/\sigma_{cr,o} \tag{B2}$$

and

$$k' = Lx/2Ly = k/2$$

The equation determining the buckling stress can be written as

$$A_s(s/2)^4 + (k'n)^2\left[2(s/2)^2 + (k'n)^2 - \lambda'\right]\left[A_s - \frac{6}{(s/2)^2\pi^2}\left(A_2 + A_4 + A_6 + \ldots\right)\right] = 0 \tag{B3}$$

where $s = 2, 4, 6, \ldots$.

Values of the coefficient $\lambda' = 4C$ in the formula for the buckling stress as obtained from equation (B3) are plotted in figure 9. The minimum value of λ' is approximately 5.32.

APPENDIX C

THE GALERKIN METHOD

As the differential equations (1) to (3) were derived in reference 1 from the expression for the total potential by the variational process, they express the requirements of equilibrium that the forces corresponding to the u, v, and w displacements must vanish. The dimension of each term in the equations is force per area since the variation of the total potential was divided by the variation of the displacement and by an area.

If the first of the three equations, the one obtained by varying the u displacements, is represented symbolically as

$$Q(u) + R(v) + S(w) = 0 \tag{C1}$$

and an approximate solution is assumed in the form

$$u = \sum_{i=o}^{r} \sum_{j=o}^{r} a_{ij} f_{1i}(x) g_{1j}(y) \tag{C2a}$$

$$v = \sum_{i=o}^{s} \sum_{j=o}^{s} b_{ij} f_{2i}(x) g_{2j}(y) \tag{C2b}$$

$$w = \sum_{i=o}^{t} \sum_{j=o}^{t} c_{ij} f_{3i}(x) g_{3j}(y) \tag{C2c}$$

substitution of these sums in the differential equation does not, as a rule, result in a vanishing left-hand member. If the value of the left-hand member after the substitution is denoted $e(x,y)$ to designate the error, one may write

$$Q\left(\sum \sum a_{ij} f_{1i} g_{1j}\right) + R\left(\sum \sum b_{ij} f_{2i} g_{2j}\right) + S\left(\sum \sum c_{ij} f_{3i} g_{3j}\right) = e(x,y) \tag{C3}$$

The error $e(x,y)$, which is a function of the coordinates x and y, represents the amount by which the component corresponding to u of the resultant of all the external and internal forces differs from zero at any point x,y of the plate. If $e(x,y)$ were identically zero, as required by the condition of equilibrium, the work done by it would vanish for any arbitrary displacement. The equilibrium condition can be approximated if the coefficients a_{ij}, b_{ij}, and c_{ij} of the series are determined from the requirement that the work done by $e(x,y)$ must vanish for a number of virtual displacements. As equation (C1) expresses the condition of equilibrium of the forces corresponding to the u displacements, any of the displacement types represented by

$$u_{ij} = f_{1i}g_{1j} \tag{C4}$$

can be chosen as a virtual displacement provided u_{ij} does not violate the geometric constraints. The virtual work during this displacement is, therefore,

$$W = \iint f_{1i}g_{1j}\left[Q\left(\sum\sum a_{ij}f_{1i}g_{1j}\right) + R\left(\sum\sum b_{ij}f_{2i}g_{2j}\right) + S\left(\sum\sum c_{ij}f_{3i}g_{3j}\right)\right] dx\, dy \tag{C5}$$

and one condition of equilibrium is

$$W = 0 \tag{C6}$$

in agreement with the principle of virtual displacements.

In the Galerkin process as described in references 8 and 9, the functions f and g are chosen in such a manner that each product f_ig_j satisfies all the boundary conditions although, as a rule, none of them satisfies the differential equations. Hence u_{ij} in equation (C4) is a suitable virtual displacement. The requirement that the virtual work vanish for the r^2 displacement patterns contained in equation (C2a) furnishes, therefore r^2 conditions for the determination of the unknown coefficients. If it is further stipulated that the components of the forces corresponding to the v and w displacements do zero work during the s^2 and t^2 virtual displacements contained in equations (C2b) and (C2c), respectively, a total of $r^2 + s^2 + t^2$ conditions are available for the determination of the $r^2 + s^2 + t^2$ unknown coefficients in equations (C2). These conditions form a set of linear equations in the coefficients since equations (1) to (3) are linear.

When r, s, and t are increased beyond all limits and the set of equations $f_i g_j$ are complete, the virtual work vanishes for any arbitrary displacement and thus the differential equations are satisfied rigorously.

APPENDIX D

ORTHOGONAL FUNCTIONS USED IN GALERKIN'S METHOD

In the application of the Rayleigh-Ritz or Galerkin method to the solution of buckling problems it is desirable to represent the deflected shape by a linear set of admissible functions, which should preferably form a complete system of orthogonal functions. In the present problem the functions $H_n(\xi)$ can be chosen to be the normal modes of vibration of a uniform beam clamped at both ends. Such functions were used successfully in references 10 and 11.

For simplicity consider the beam to be of length 2 and clamped at $\xi = -1$ and 1. These functions in the normalized form can be written for an odd number of waves as

$$\varphi_m(\xi) = c_m \cos k_m\xi - d_m \cosh k_m\xi \tag{D1}$$

where

$$c_m = \frac{\cosh k_m}{\left(\cos^2 k_m + \cosh^2 k_m\right)^{1/2}}$$

$$d_m = \frac{\cos k_m}{\left(\cos^2 k_m + \cosh^2 k_m\right)^{1/2}}$$

and k_m, as determined from the equation

$$\tan k_m + \tanh k_m = 0 \tag{D2}$$

has the values

$$k_1 = 2.3650204$$

$$k_3 = 5.4978$$

$$k_5 = 8.6394$$

For an even number of waves, the normalized functions are

$$\varphi_n = g_n \sinh k_n\xi - h_n \sin k_n\xi \tag{D3}$$

The meaning of the symbols is

$$g_n = \frac{\sin k_n}{\left(\sin^2 k_n + \sinh^2 k_n\right)^{1/2}}$$

$$h_n = \frac{\sinh k_n}{\left(\sin^2 k_n + \sinh^2 k_n\right)^{1/2}}$$

and k_n, as determined from the equation

$$\tan k_n - \tanh k_n = 0 \tag{D4}$$

has the values

$$k_2 = 3.9266$$

$$k_4 = 7.0686$$

$$k_6 = 10.2102$$

It can be easily verified that

$$\int_{-1}^{1} \varphi_m(\xi)\varphi_n(\xi)\, d\xi = \delta_{mn} \tag{D5}$$

where

$$\delta_{mn} = 1 \qquad \text{when} \quad m = n$$

$$\delta_{mn} = 0 \qquad \text{when} \quad m \neq n$$

moreover

$$\int_{-1}^{1} \varphi_{2m+1}''(\xi)\varphi_{2n+2}(\xi)\, d\xi = 0 \qquad m,n = 0, 1, 2, \ldots \qquad \text{(D6)}$$

$$\int_{-1}^{1} \varphi_{2n+2}(\xi)\varphi_{2m+1}(\xi)\, d\xi = 0 \qquad m,n = 0, 1, 2, \ldots \qquad \text{(D7)}$$

$$\int_{-1}^{1} \varphi_m(\xi)\varphi_n''(\xi)\, d\xi = \gamma_{nm} = \int_{-1}^{1} \varphi_m(\xi)\varphi_m''(\xi)\, d\xi = \gamma_{mn} \qquad \text{(D8)}$$

$$\varphi_s'''(1) = \left(k_s \tanh k_s\right)\varphi_s''(1) \qquad \text{(D9)}$$

In the numerical calculations the following constants were needed:

$\tanh k_1 = 0.98250295$ $\qquad$ $\tanh k_3 = 0.99996645$

$\varphi_1''(1) = 7.8407388$ $\qquad$ $\varphi_3''(1) = -42.7448996$

$\gamma_{11} = -3.0754792$ $\qquad$ $\gamma_{13} = -2.43256274$

$\gamma_{33} = -24.7262334$

APPENDIX E

SUMMATION OF SERIES BY COMPLEX INTEGRATION

The series appearing in the solution by Galerkin's method can be reduced to the following two types:

$$\sum_{m=1}^{\infty} \frac{1}{m^2 - a^2} \qquad \sum_{m=1}^{\infty} \frac{1}{\left(m^2 - a^2\right)^2}$$

where a may be real (nonintegral) or pure imaginary. The sums of these series can be found with the aid of complex integration.

In the evaluation of the first series let $f(z) = 1/\left(z^2 - a^2\right)$. The two poles of this function are $z = \pm a$ with residues of value $\pm(1/2a)$. Hence, according to pages 133 to 135 of reference 12:

$$\sum_{m=1}^{\infty} \frac{1}{m^2 - a^2} = -\frac{1}{2}\left(\frac{\pi}{a} \cot \pi a - \frac{1}{a^2}\right) \tag{E1}$$

similarly,

$$\sum_{m=1}^{\infty} \frac{1}{m^2 + a^2} = \frac{1}{2}\left(\frac{\pi}{a} \coth \pi a - \frac{1}{a^2}\right) \tag{E2}$$

In the evaluation of the second series let $f(z) = 1/\left(z^2 - a^2\right)^2$ which has two double poles at $z = \pm a$. Now consider the function $\pi \cot \pi z f(z)$. The residues at $z = \pm a$ are

$$\frac{d}{dz}\left[\frac{(z \pm a)^2 \pi \cot \pi z}{\left(z^2 - a^2\right)^2}\right]_{z=a} = \frac{d}{dz}\left[\frac{\pi \cot \pi z}{(z \pm a)^2}\right]_{z=a}$$

$$= -\left(\frac{\pi^2}{4a^2} \frac{1}{\sin^2 \pi a} + \frac{2\pi \cot \pi a}{8a^3}\right)$$

Hence

$$\sum_{m}^{\infty} \frac{1}{\left(m^2 - a^2\right)^2} = \frac{\pi^2}{4a^2}\left(\frac{1}{\sin^2 \pi a} + \frac{1}{\pi a} \cot \pi a\right) - \frac{1}{2a^4} \tag{E3}$$

and similarly

$$\sum_{m}^{\infty} \frac{1}{\left(m^2 + a^2\right)^2} = \frac{\pi^2}{4a^2}\left(\frac{1}{\sinh^2 \pi a} + \frac{1}{\pi a} \coth \pi a\right) - \frac{1}{2a^4} \tag{E4}$$

The sum of the following series can be found by first reducing the fraction to partial fractions:

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a_o^2\right)\left(m^2 - t_o^2\right)\left(m^2 - n_o^2\right)} = \sum_{m=1}^{\infty} \frac{c_1}{m^2 + a_o^2} + \sum_{m=1}^{\infty} \frac{c_2}{m^2 - t_o^2} + \sum_{m=1}^{\infty} \frac{c_3}{m^2 - n_o^2}$$

where

$$t_o = \frac{2t - 1}{2} \qquad n_o = \frac{2n - 1}{2} \qquad t,n = 1, 2, \ldots$$

It can be shown that:

$$c_1 = -\frac{a_o^2}{\left(a_o^2 + t_o^2\right)\left(a_o^2 + n_o^2\right)}$$

$$c_2 = \frac{t_o^2}{\left(a_o^2 + t_o^2\right)\left(t_o^2 - n_o^2\right)}$$

$$c_3 = \frac{n_o^2}{\left(a_o^2 + n_o^2\right)\left(n_o^2 - t_o^2\right)}$$

Hence with $\cot \pi t_o = \cot \pi n_o = 0$

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a_o^2\right)\left(m^2 - t_o^2\right)\left(m^2 - n_o^2\right)} = -\frac{\pi a_o \coth \pi a_o}{2\left(a_o^2 + t_o^2\right)\left(a_o^2 + n_o^2\right)} \tag{E5}$$

when $t_o \neq n_o$.

When $t_o = n_o$ the above partial fraction expansion has to be modified in the following manner:

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a_o^2\right)\left(m^2 - t_o^2\right)^2} = \sum_{m=1}^{\infty} \frac{c_1}{m^2 + a_o^2} + \sum_{m=1}^{\infty} \frac{c_2}{\left(m^2 - t_o^2\right)^2} + \sum_{m=1}^{\infty} \frac{c_3}{m^2 - t_o^2}$$

where

$$c_1 = -\frac{a_o^2}{\left(a_o^2 + t_o^2\right)^2}$$

$$c_2 = \frac{t_o^2}{a_o^2 + t_o^2}$$

$$c_3 = \frac{a_o^2}{\left(a_o^2 + t_o^2\right)^2}$$

Because of equations (E1), (E2), and (E3) the result is

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a_o^2\right)\left(m^2 - t_o^2\right)^2} = -\frac{\pi a_o \coth \pi a_o}{2\left(a_o^2 + t_o^2\right)^2} + \frac{\pi^2}{4\left(a_o^2 + t_o^2\right)} \tag{E6}$$

In a similar manner:

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a_o^2\right)\left(m^2 - t_o^2\right)\left(m^4 - k_{so}^4\right)} = \frac{\pi}{2}\left\{\frac{a_o \coth \pi a_o}{\left(a_o^2 + t_o^2\right)\left(a_o^4 - k_{so}^4\right)} - \right.$$

$$\left.\frac{1}{2k_{so}}\left[\frac{\cot \pi k_{so}}{\left(a_o^2 + k_{so}^2\right)\left(k_{so}^2 - t_o^2\right)} + \frac{\coth \pi k_{so}}{\left(a_o^2 - k_{so}^4\right)\left(k_{so}^2 + t_o^2\right)}\right]\right\}$$

$$= \frac{\pi}{2} \frac{1}{\left(a_o^4 - k_{so}^4\right)}\left[\frac{a_o \coth \pi a_o}{\left(a_o^2 + t_o^2\right)} - \frac{\coth \pi k_{so}}{k_{so}} \frac{\left(a_o^2 t_o^2 - k_{so}^4\right)}{\left(t_o^4 - k_{so}^4\right)}\right] \tag{E7}$$

In the above expression and in expressions where k_{so} appears, the relation previously obtained in appendix D, that is,

$$\tan \pi k_{so} + \tanh \pi k_{so} = 0 \tag{D2}$$

is used to simplify the summations as follows:

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a_o^2\right)\left(m^4 - k_{to}^4\right)\left(m^4 - k_{so}^4\right)} = \frac{\pi}{2}\left\{\frac{-a_o \coth \pi a_o}{\left(a_o^4 - k_{to}^4\right)\left(a_o^4 - k_{so}^4\right)} + \right.$$

$$\left.\frac{a_o^2}{\left(k_{so}^4 - k_{to}^4\right)}\left[\frac{\coth \pi k_{to}}{k_{to}\left(a_o^4 - k_{to}^4\right)} - \frac{\coth \pi k_{so}}{k_{so}\left(a_o^4 - k_{so}^4\right)}\right]\right\} \qquad \text{when} \quad s \neq t \tag{E8}$$

$$= \frac{\pi}{2}\left[-\frac{a_o \coth \pi a_o}{\left(a_o^4 - k_{to}^4\right)} + \frac{\pi}{4k_{to}^4\left(a_o^2 - k_{to}^2\right)} - \right.$$

$$\left.\frac{a_o^2 \coth \pi k_{to}\left(a_o^4 - 5k_{to}^4\right)}{4k_{to}^5\left(a_o^4 - k_{to}^4\right)^2} - \frac{\pi \operatorname{cosec}^2 \pi k_{to}}{4k_{to}^2\left(a_o^4 - k_{to}^4\right)}\right] \qquad \text{when} \quad s = t \tag{E9}$$

The following two series can be summed up in a similar manner:

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a^2\right)\left(m^2 + b^2\right)\left(m^2 + c^2\right)} = -\frac{\pi}{2}\left[\frac{a \coth \pi a}{\left(a^2 - b^2\right)\left(a^2 - c^2\right)} + \frac{b \coth \pi b}{\left(b^2 - a^2\right)\left(b^2 - c^2\right)} + \frac{c \coth \pi c}{\left(c^2 - a^2\right)\left(c^2 - b^2\right)}\right] \tag{E10}$$

$$\sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a^2\right)\left(m^2 + b^2\right)\left(m^4 - k_{so}{}^4\right)} = \frac{\pi}{2}\left\{\frac{1}{\left(a^2 - b^2\right)}\left[\frac{a \coth \pi a}{\left(a^4 - k_{so}{}^4\right)} - \frac{b \coth \pi b}{\left(b^4 - k_{so}{}^4\right)} + \frac{\coth \pi k_{so}\left(a^2 b^2 + k_{so}{}^4\right)}{k_{so}\left(a^4 - k_{so}{}^4\right)\left(b^4 - k_{so}{}^4\right)}\right]\right\} \tag{E11}$$

In order to find the sum of the series

$$\sum_{t=1}^{\infty} \frac{t_o{}^2}{\left(t_o{}^2 + a_o{}^2\right)\left(t_o{}^2 + b_o{}^2\right)\left(t_o{}^2 + d_o{}^2\right)}$$

where $t_o = (2t - 1)/2$, let $\bar{t} = 2t - 1$, so that

$$\sum_{t=1}^{\infty} \frac{t_o{}^2}{\left(t_o{}^2 + a_o{}^2\right)\left(t_o{}^2 + b_o{}^2\right)\left(t_o{}^2 + d_o{}^2\right)} = 16 \sum_{\bar{t}=1,3,5,\ldots}^{\infty} \frac{\bar{t}^2}{\left[\bar{t}^2 + \left(2a_o\right)^2\right]\left[\bar{t}^2 + \left(2b_o\right)^2\right]\left[\bar{t}^2 + \left(2d_o\right)^2\right]}$$

Now consider:

$$\sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{\left(m^2 + a^2\right)\left(m^2 + b^2\right)\left(m^2 + c^2\right)} = \sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a^2\right)\left(m^2 + b^2\right)\left(m^2 + c^2\right)} -$$

$$\sum_{m=1}^{\infty} \frac{(2m)^2 \left[(2m)^2 + c^2\right]^{-1}}{\left[(2m)^2 + a^2\right]\left[(2m)^2 + b^2\right]}$$

$$= \sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 + a^2\right)\left(m^2 + b^2\right)\left(m^2 + c^2\right)} - \frac{1}{16}\sum_{m=1}^{\infty} \frac{m^2}{\left[m^2 + (a/2)^2\right]\left[m^2 + (b/2)^2\right]\left[m^2 + (c/2)^2\right]}$$

$$= -\frac{\pi}{2}\left[\frac{a}{\left(a^2 - b^2\right)\left(a^2 - c^2\right)}\left(\coth \pi a - \frac{1}{2}\coth\frac{\pi a}{2}\right) + \frac{b}{\left(b^2 - a^2\right)\left(b^2 - c^2\right)}\left(\coth \pi b - \right.\right.$$

$$\left.\left.\frac{1}{2}\coth\frac{\pi b}{2}\right) + \frac{c}{\left(c^2 - a^2\right)\left(c^2 - b^2\right)}\left(\coth \pi c - \frac{1}{2}\coth\frac{\pi c}{2}\right)\right]$$

$$= -\frac{\pi}{4}\left[\frac{a \tanh (\pi a/2)}{\left(a^2 - b^2\right)\left(a^2 - c^2\right)} + \frac{b \tanh (\pi b/2)}{\left(b^2 - a^2\right)\left(b^2 - c^2\right)} + \frac{c \tanh (\pi c/2)}{\left(c^2 - a^2\right)\left(c^2 - b^2\right)}\right]$$

Hence

$$\sum_{t=1}^{\infty} \frac{t_o^2}{\left(t_o^2 + a_o^2\right)\left(t_o^2 + b_o^2\right)\left(t_o^2 + d_o^2\right)} = -\frac{\pi}{2}\left[\frac{a_o \tanh \pi a_o}{\left(a_o^2 - b_o^2\right)\left(a_o^2 - d_o^2\right)} + \frac{b_o \tanh \pi b_o}{\left(b_o^2 - a_o^2\right)\left(b_o^2 - d_o^2\right)} + \frac{d_o \tanh \pi d_o}{\left(d_o^2 - a_o^2\right)\left(d_o^2 - b_o^2\right)}\right] \tag{E12}$$

observing that

$$a_o^2 - b_o^2 = -\frac{(1 + \mu)}{(1 - \mu)} \frac{a_2}{4} \qquad b_o^2 - d_o^2 = \frac{(1 + \mu)}{(1 - \mu)} \frac{\lambda_1}{4}$$

$$d_o^2 - a_o^2 = \frac{(1 + \mu)}{8} (kn)^2$$

and inserting these values into equation (E12) one obtains

$$\sum_{t=1}^{\infty} \frac{t_o^2}{\left(t_o^2 + a_o^2\right)\left(t_o^2 + b_o^2\right)\left(t_o^2 + d_o^2\right)} = -\frac{4\pi(1 - \mu)}{(1 + \mu)^2(kn)^2}\left[\frac{\tanh \pi a_o}{a_o} + \frac{(kn)^2 \tanh \pi b_o}{b_o \lambda_1} - \frac{4d_o \tanh \pi d_o}{\lambda_1}\right] \tag{E13}$$

Similarly:

$$\sum_{t=1}^{\infty} \frac{t_o^2}{\left(t_o^2 + a_o^2\right)\left(t_o^2 + d_o^2\right)\left(t_o^4 - k_{so}^4\right)} = \frac{\pi}{2}\left[\frac{1}{\left(a_o^2 - d_o^2\right)}\left(\frac{a_o \tanh \pi a_o}{a_o^4 - k_{so}^4} - \frac{d_o \tanh \pi d_o}{\left(d_o^4 - k_{so}^4\right)}\right) + \frac{k_{so}\left(a_o^2 + d_o^2\right) \tanh \pi k_{so}}{\left(a_o^4 - k_{so}^4\right)\left(d_o^4 - k_{so}^4\right)}\right] \tag{E14}$$

which can be written as

$$\sum_{t=1}^{\infty} \frac{t_o^2}{\left(t_o^2 + a_o^2\right)\left(t_o^2 + d_o^2\right)\left(t_o^4 - k_{so}^4\right)} = \frac{4\pi}{(1 + \mu)(kn)^2}\left[\frac{d_o \tanh \pi d_o}{d_o^4 - k_{so}^4} - \right.$$

$$\left.\frac{a_o \tanh \pi a_o}{\left(a_o^4 - k_{so}^4\right)} + \frac{(1 + \mu)(kn)^2}{8} \frac{\left(a_o^2 + d_o^2\right)k_{so} \tanh \pi k_{so}}{\left(a_o^4 - k_{so}^4\right)\left(d_o^4 - k_{so}^4\right)}\right] \tag{E15}$$

$$\sum_{t=1}^{\infty} \frac{t_o^4}{\left(t_o^2 + b_o^2\right)\left(t_o^2 + d_o^2\right)\left(t_o^4 - k_{so}^4\right)} = \frac{\pi}{2}\left[\frac{1}{\left(b_o^2 - d_o^2\right)}\left(- \frac{b_o^3 \tanh \pi b_o}{b_o^4 - k_{so}^4} + \right.\right.$$

$$\left.\left.\frac{d_o^3 \tanh \pi d_o}{\left(d_o^4 - k_{so}^4\right)}\right) - \frac{k_{so} \tanh \pi k_{so}\left(b_o^2 d_o^2 + k_{so}^4\right)}{\left(b_o^4 - k_{so}^4\right)\left(d_o^4 - k_{so}^4\right)}\right] \tag{E16}$$

The following sums can be evaluated in a similar manner:

$$\sum_{t=1}^{\infty} \frac{t_o^2}{\left(t_o^2 + d_o^2\right)\left(t_o^4 - k_{so}^4\right)\left(t_o^4 - k_{to}^4\right)} = \frac{\pi}{2}\left[- \frac{d_o \tanh \pi d_o}{\left(d_o^4 - k_{so}^4\right)\left(d_o^4 - k_{to}^4\right)} - \right.$$

$$\left.\frac{k_{so} \tanh \pi k_{so}}{\left(d_o^4 - k_{so}^4\right)\left(k_{so}^4 - k_{to}^4\right)} - \frac{k_{to} \tanh \pi k_{to}}{\left(d_o^4 - k_{to}^4\right)\left(k_{to}^4 - k_{so}^4\right)}\right] \quad \text{when} \quad s \neq t \tag{E17}$$

$$= \frac{\pi}{2}\left[- \frac{d_o \tanh \pi d_o}{\left(d_o^4 - k_{to}^4\right)^2} + \frac{\pi}{4k_{to}^4\left(d_o^2 - k_{to}^2\right)} + \frac{\tanh \pi k_{to}\left(d_o^4 + 3k_{to}^4\right)}{4k_{to}^3\left(d_o^4 - k_{to}^4\right)^2} - \right.$$

$$\left.\frac{\pi \sec^2 \pi k_{to}}{4k_{to}^2\left(d_o^4 - k_{to}^4\right)}\right] \quad \text{when} \quad s = t \tag{E18}$$

$$\sum_{t=1}^{\infty} \frac{t_o^4}{\left(t_o^2 + d_o^2\right)\left(t_o^4 - k_{so}^4\right)\left(t_o^4 - k_{to}^4\right)} = \frac{\pi}{2}\left[\frac{d_o^3 \tanh \pi d_o}{\left(d_o^4 - k_{so}^4\right)\left(d_o^4 - k_{to}^4\right)} + \right.$$

$$\left. \frac{k_{so} \tanh \pi k_{so}}{\left(d_o^4 - k_{so}^4\right)\left(k_{so}^4 - k_{to}^4\right)} + \frac{k_{to} \tanh \pi k_{to}}{\left(d_o^4 - k_{to}^4\right)\left(k_{to}^4 - k_{so}^4\right)}\right] \quad \text{when} \quad s \neq t \tag{E19}$$

$$= \frac{\pi}{2}\left[\frac{d_o^3 \tanh \pi d_o}{\left(d_o^4 - k_{to}^4\right)^2} - \frac{\pi}{4k_{to}^2\left(d_o^2 - k_{to}^2\right)} - \frac{\tanh \pi k_{to}\left(d_o^4 + 3k_{to}^4\right)d_o^2}{4k_{to}^3\left(d_o^4 - k_{to}^4\right)^2} + \right.$$

$$\left. \frac{\pi d_o^2 \sec^2 \pi k_{to}}{4k_{to}^2\left(d_o^4 - k_{to}^4\right)}\right] \quad \text{when} \quad s = t \tag{E20}$$

It may be added here that in deriving equation (C22) from equation (C21), the following relations were needed:

$$\left[2a_o^4 - (1 - \mu)k_{to}^4\right] + (1 + \mu)a_o^2 b_o^2 = (1 - \mu)\left(b_o^4 - k_{to}^4\right) \tag{E21}$$

$$\left[2a_o^4 - (1 - \mu)k_{to}^4\right] + (1 + \mu)a_o^2 d_o^2 = (1 - \mu)\left(d_o^4 - k_{to}^4\right) + \left[(1 + \mu)\lambda_1/4\right]\left(a_o^2 + d_o^2\right) \tag{E22}$$

$$\left[2a_o^4 - (1 - \mu)k_{to}^4\right]\left(b_o^2 + d_o^2\right) + (1 + \mu)a_o^2\left(b_o^2 d_o^2 + k_{to}^4\right) = (1 - \mu)\left(a_o^2 + d_o^2\right)\left(b_o^4 - k_{to}^4\right) \tag{E23}$$

$$\left[2a_o^4 - (1 - \mu)k_{to}^4\right] + (1 + \mu)a_o^2 k_{to}^2 = (1 - \mu)\left(a_o^2 + k_{to}^2\right)\left(b_o^2 - k_{to}^2\right) \tag{E24}$$

REFERENCES

1. Hoff, N. J.: Bending and Buckling of Rectangular Sandwich Plates. NACA TN 2225, 1950.

2. Leggett, D. M. A.: The Elastic Stability of a Long and Slightly Bent Rectangular Plate under Uniform Shear. Proc. Roy. Soc. (London), ser. A, vol. 162, no. 908, Sept. 1, 1937, pp. 62-83.

3. Smith, R. C. T.: The Buckling of Flat Plywood Plates in Compression. Rep. ACA-12, Australian Council for Aeronautics, Dec. 1944.

4. Boller, K. H.: Buckling Loads of Flat Sandwich Panels in Compression. Buckling of Flat Sandwich Panels with Loaded Edges Simply Supported and the Remaining Edges Clamped. Rep. No. 1525-B, Forest Products Lab., U. S. Dept. Agric., Sept. 1947.

5. Reddick, H. W., and Miller, F. H.: Advanced Mathematics for Engineers. John Wiley & Sons, Inc., 1938.

6. Courant, R., and Hilbert, D.: Methoden der mathematischen Physik. Vol. 1. Julius Springer (Berlin), 1931, p. 353.

7. Timoshenko, S.: Theory of Plates and Shells. McGraw-Hill Book Co., Inc., 1940.

8. Duncan, W. J.: Galerkin's Method in Mechanics and Differential Equations. R. & M. No. 1798, British A.R.C., 1937.

9. Duncan, W. J.: The Principles of the Galerkin Method. R. & M. No. 1848, British A.R.C., 1938.

10. Timoshenko, S.: Vibration Problems in Engineering. Second ed., D. Van Nostrand Co., Inc., 1937.

11. Maulbetsch, J. L.: Buckling of Compressed Rectangular Plates with Built-In Edges. Jour. Appl. Mech., vol. 4, no. 2, June 1937, pp. 59-62.

12. Phillips, E. G.: Functions of a Complex Variable. Interscience Publishers, Inc. (New York), 1946.

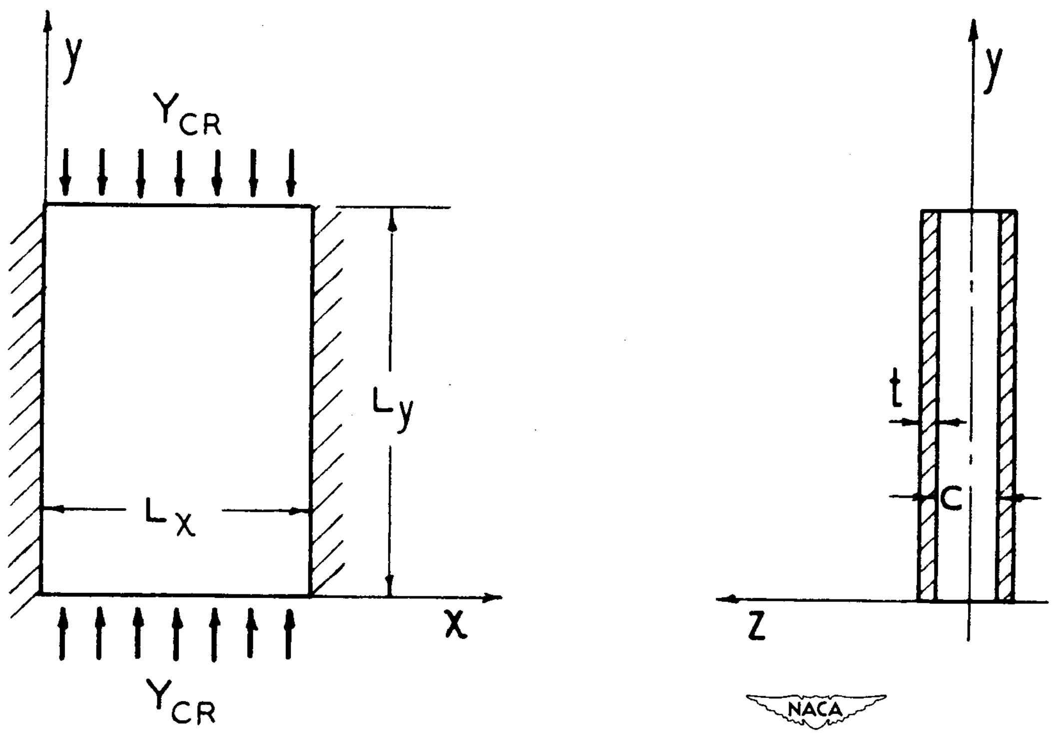
TABLE I

SAMPLE LOWER AND UPPER BOUNDS OF CRITICAL

STRESS FACTOR C_{min}

r \ R	10			0.30			0.10			0.05		
	Lower bound (a)	Upper bound (a)	Difference (percent) (b)	Lower bound (a)	Upper bound (a)	Difference (percent) (b)	Lower bound (a)	Upper bound (a)	Difference (percent) (b)	Lower bound (a)	Upper bound (a)	Difference (percent) (b)
4	1.495	1.638	9	0.538	0.572	6.32	0.239	0.248	6.32			
	True value, 1.566			True value, 0.555			True value, 0.245					
39	1.627 -----	1.694 -----	4.12 ----	1.285 [c]1.305	1.426 [c]1.407	11 7	0.957 -----	1.040 -----	8.7 ---			
	True value, 1.661			True value, 1.356			True value, 0.997					
99	1.625 -----	1.708 -----	5.11 ----	1.448 [c]1.471	1.608 [c]1.589	11.05 8.02				1.038 -----	1.130 -----	8.86 ----
	True value, 1.667			True value, 1.530						True value, 1.084		

[a]First approximation unless otherwise specified.
[b]Percentage difference is referred to lower bound.
[c]Second approximation.



EDGES $x = 0$, $x = L_x$ ARE FIXED

EDGES $y = 0$, $y = L_y$ ARE SIMPLY SUPPORTED

Figure 1.- Sandwich plate. (Thicknesses are exaggerated.)

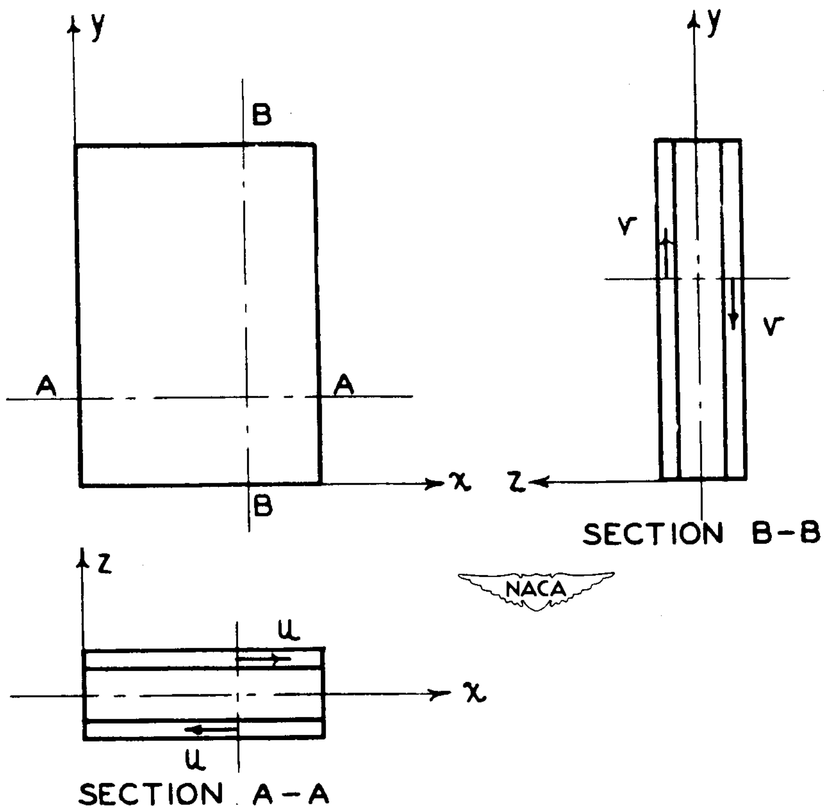


Figure 2.- Displacements in plane of plate.

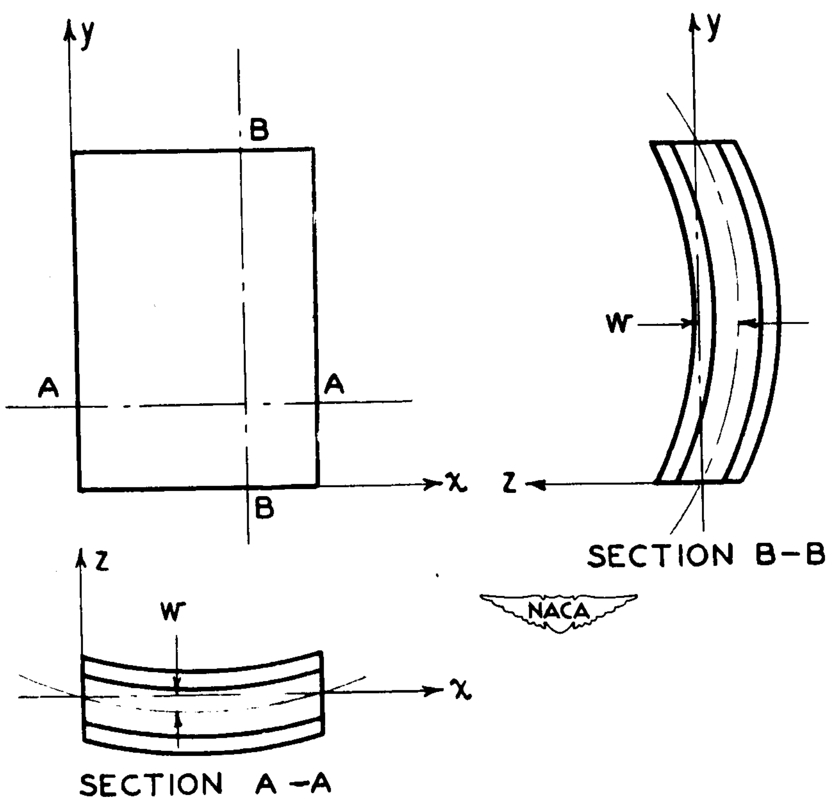


Figure 3.- Displacements out of plane of plate. (Thicknesses are exaggerated.)

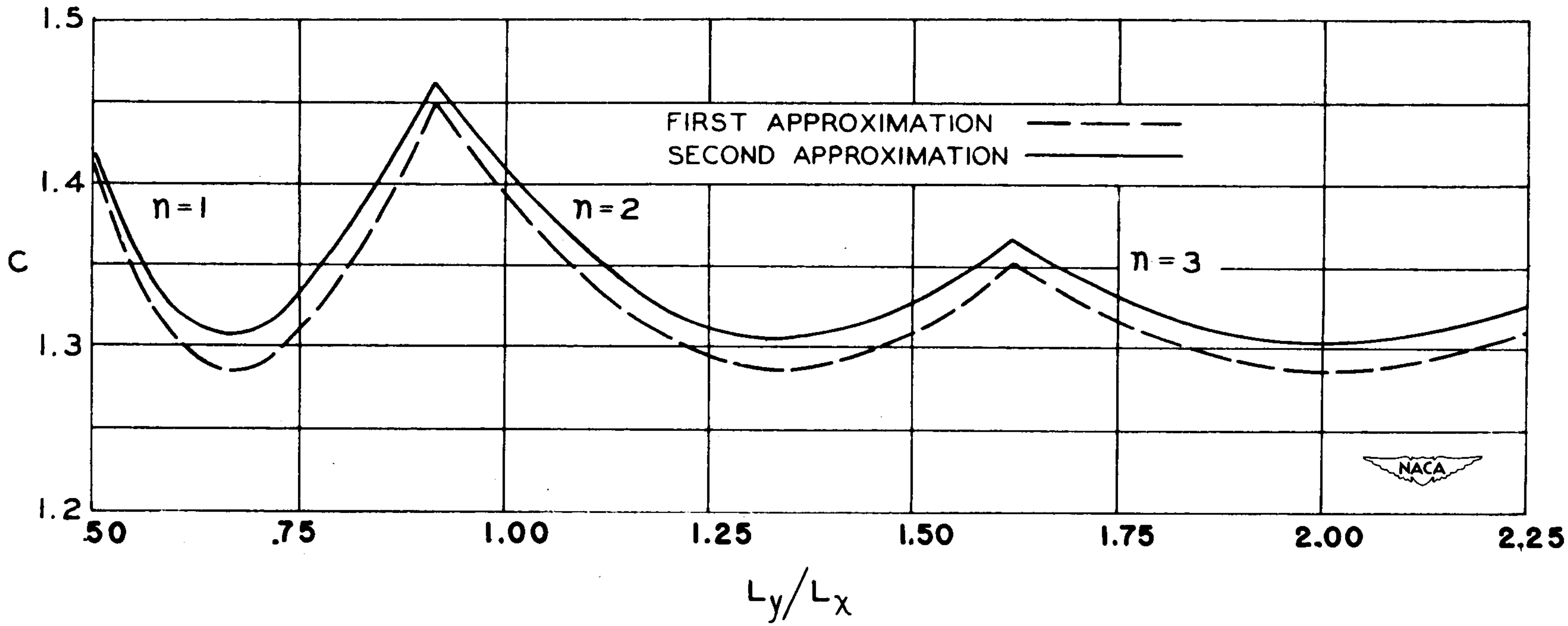


Figure 4.- Lower bound of critical stress factor by Leggett's method for $r = 39$ and $R = 0.3$.

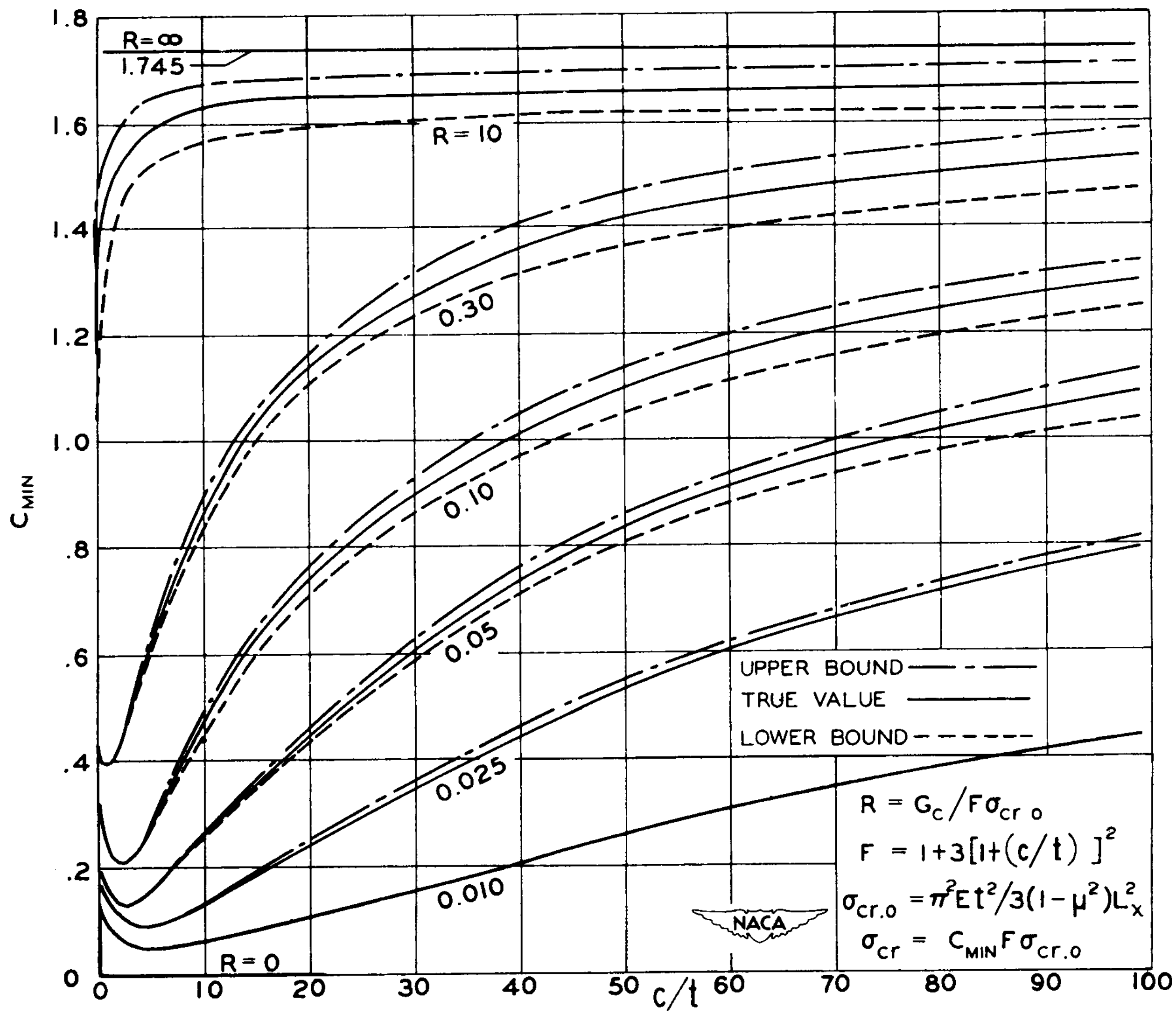


Figure 5.- Critical stress factor of rectangular sandwich panel.

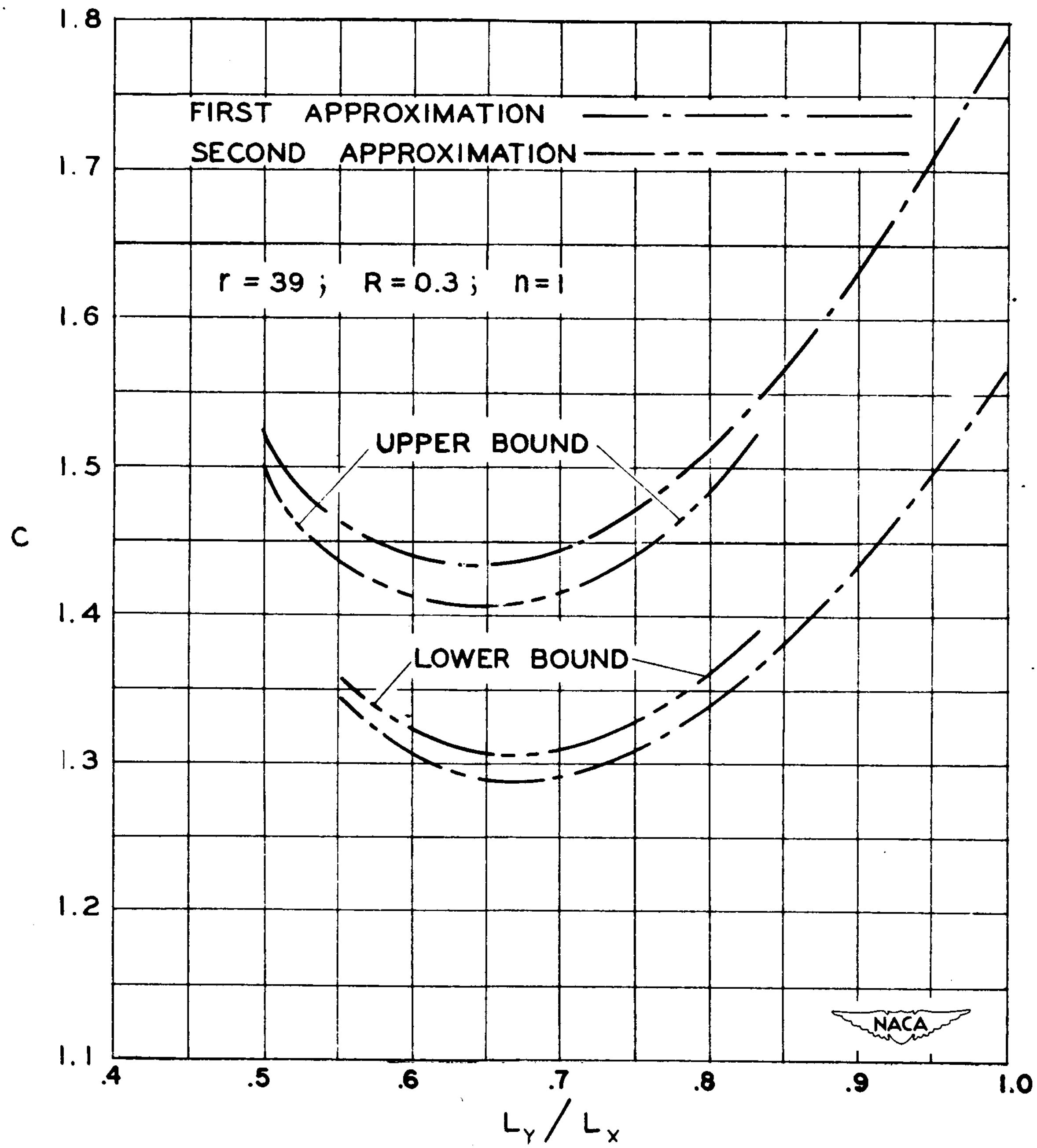


Figure 6.- Upper and lower bounds of critical stress factor for $r = 39$ and $R = 0.3$.

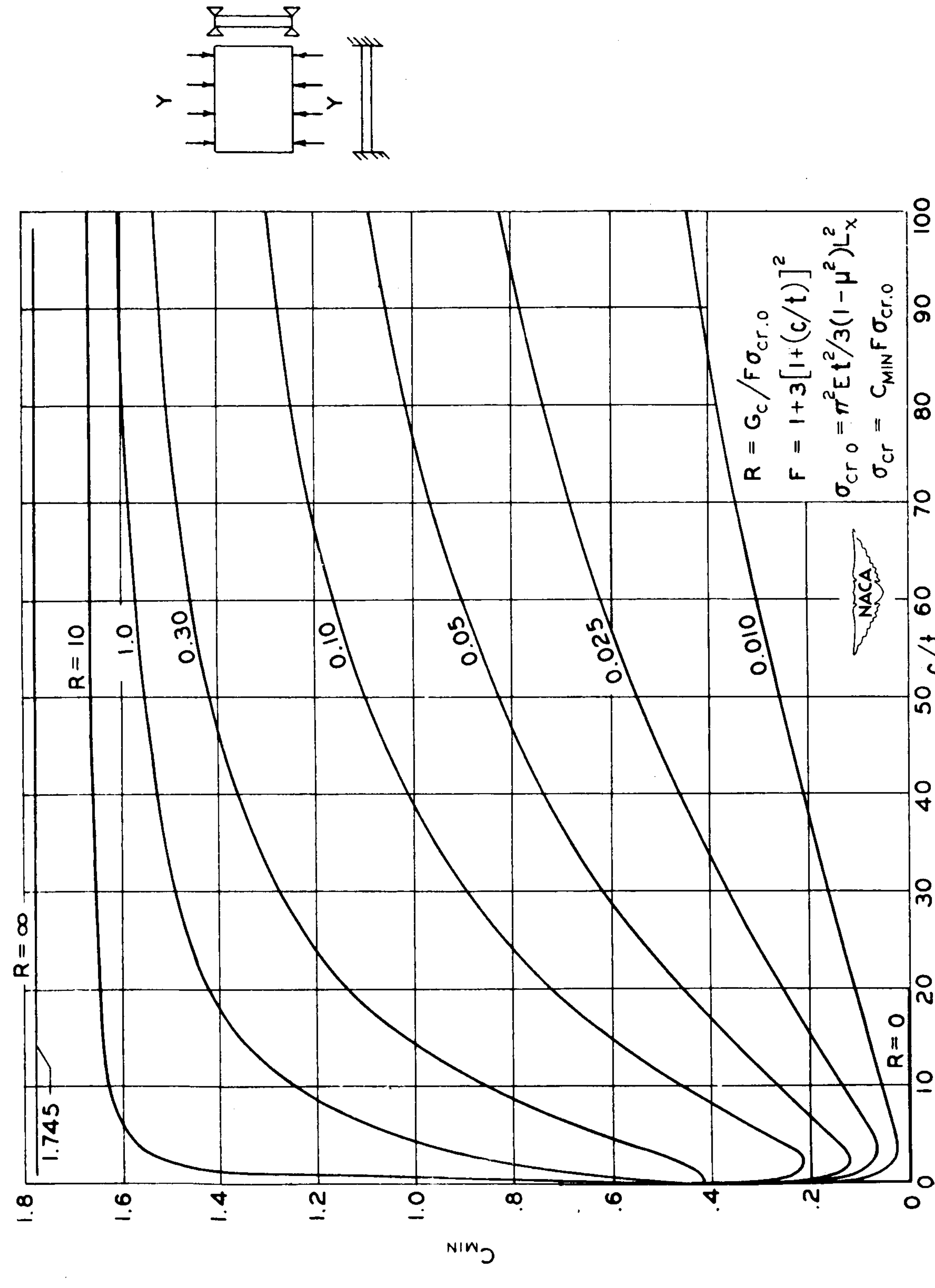


Figure 7.- True critical stress factor for rectangular sandwich plate.

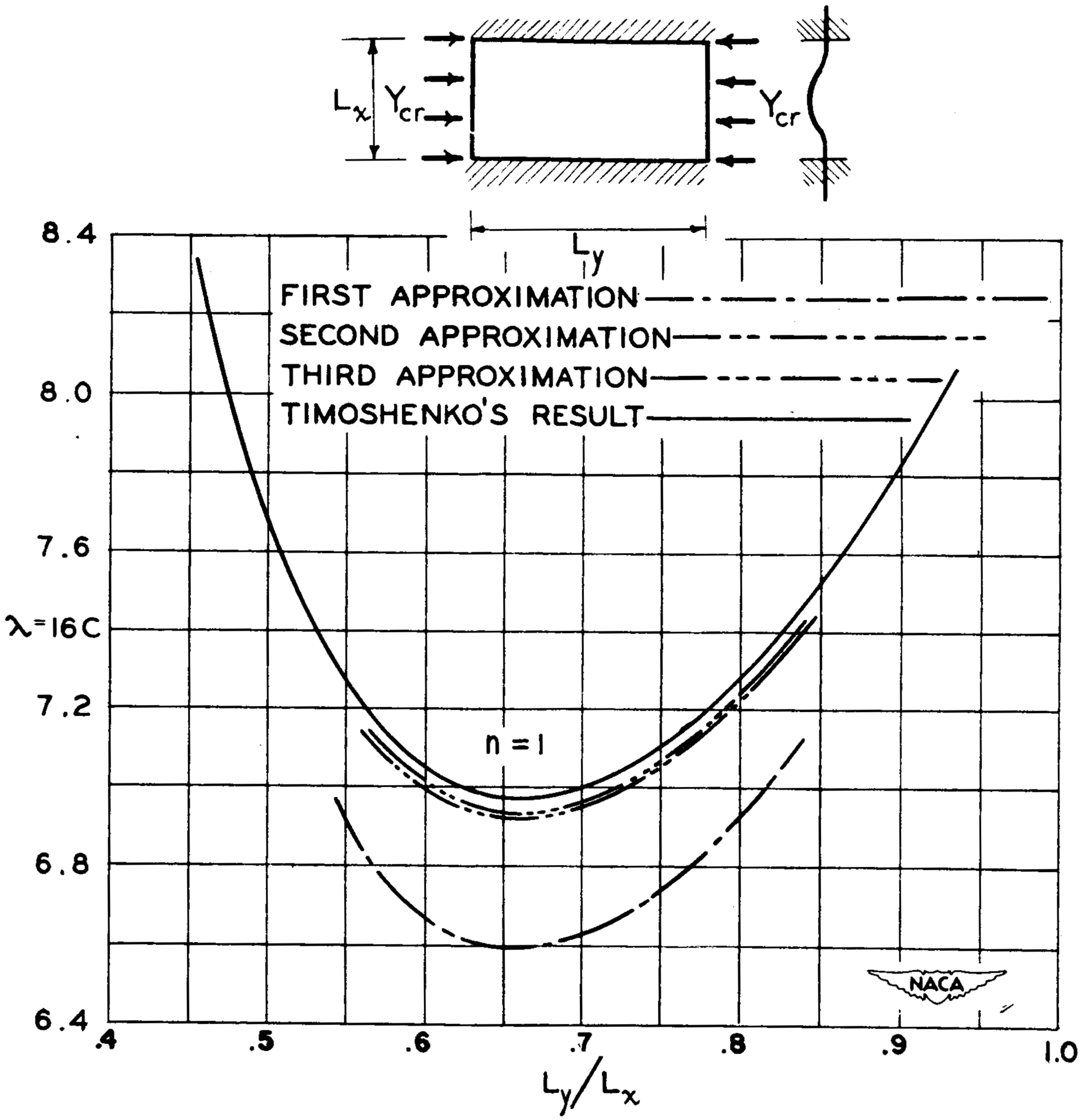


Figure 8.- Critical stress factor for a thin plate loaded as shown.

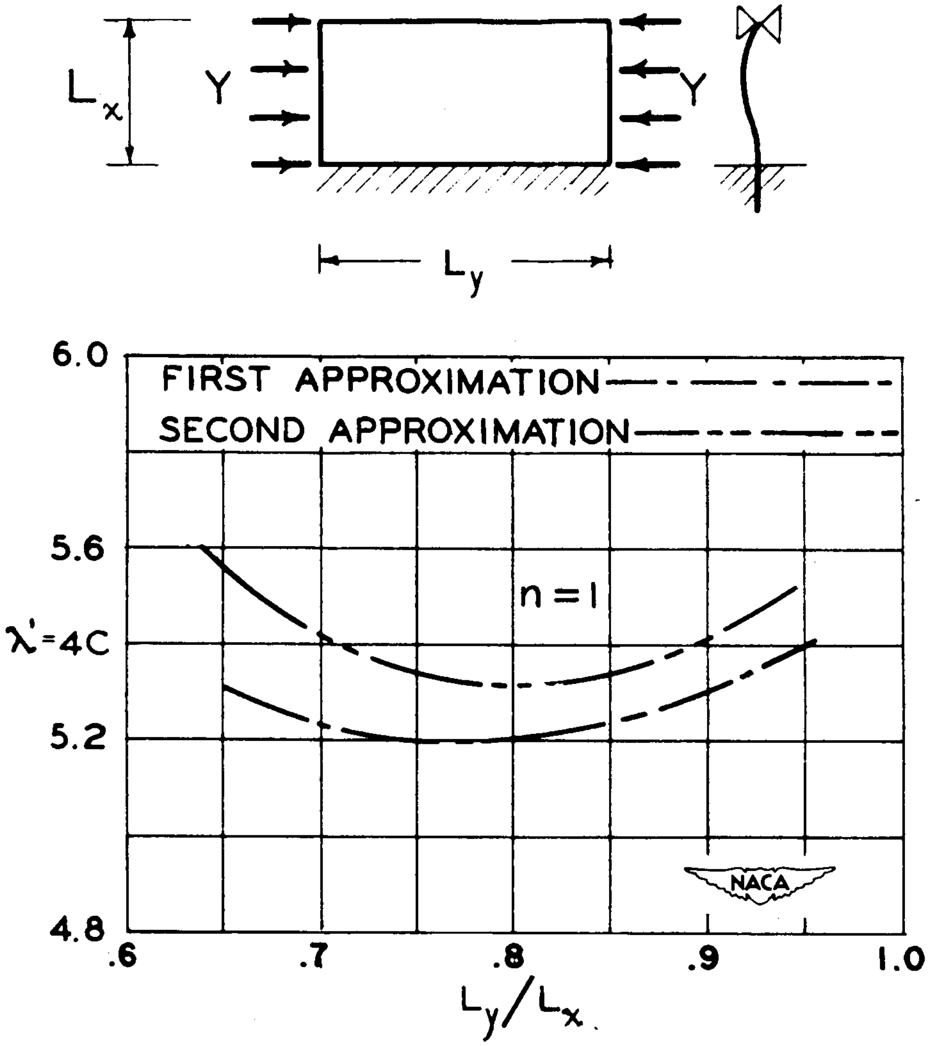


Figure 9.- Critical stress factor for a thin plate loaded as shown.